Response to Andras Komaromi

I am somewhat surprised that nobody has responded to you. While I know nothing about the paper you mention, the answer is very easy to get in several ways. Don Knuth deals with it in his exercise 1.3.2-12 in *The Art of Computer Programming*, which is getting on 40 years since first published. He naturally defines the element as a saddle point, describes a couple of efficient algorithms to find one if there is (there can be at most one, if you insist on strict inequalities), and en passant remarks that the probability of there being one in a given position is $(m+n)/[mn\binom{n+m}{m}] = B(m,n)$, where this $B()$ is the beta function.

Knuth did it from combinatorial considerations, but I prefer the analytic derivation: If the elements are all independently drawn from the completely continuous distribution F, then for an entry to have the value x, be smaller than the other $m-1$ values in its row, and exceed all of the $n-1$ other elements in its column, it has to have the conditional probability density function $r(x)$ given by

$$r(x) = f(x)F(x)^{m-1}(1-F(x))^{n-1},$$

where $f(x)$ is the density derived from F. If you integrate this over all possible x you get the probability that this event takes place,

$$\Pr[\text{saddle point}] = \int f(x)F(x)^{m-1}(1-F(x))^{n-1}dx = \int_0^1 u^{m-1}(1-u)^{n-1}du = B(m,n),$$

where the change of integration variable $F(x) \to u$ simplified the calculation.

Note that for any but negligibly small matrices this is an awfully small number...

When I worked on this issue I only did calculations assuming the most benign symmetry and independence assumptions. For all that I know interesting games display vastly different traits, and the probability of finding in them a saddle point may be far more interesting, that is, much larger.

If you or anyone else can enlighten me on this point, hofri@wpi.edu, I shall be most grateful.

Hope this helped,

—Micha Hofri