Contractual Matching: Limits of Decentralization

Joon Song*

Department of Economics, UCLA

November 27, 2005

Abstract

This paper considers *incentive-constrained* efficient contractual matching in the presence of moral hazard. Teams/firms of finite individuals form in order to produce stochastic output in the presence of moral hazard. I characterize by the duality in linear programming an idealized market environment for efficiency.

Contract arbitrageurs specializing in writing contracts for teams, insurers, a market for lotteries on contracts, Lindahl prices for the lotteries, and public randomization device are shown to be required for efficiency in the finite economy. A public randomization device needs to accommodate and signal enormous amounts of information since all the firms' random contracts are correlated. Insurers are committed to avoid cream-skimming, and essentially cross-subsidize across firms. Also, insurers need to know all the details of the economy to calculate the right prices. In the continuum economy, each firm possesses its own independent randomization device for the exercise of its contract, and the role of insurers is replaced by ex-post market, in which contract arbitrageurs trade in contingent claims market before the realization of idiosyncratic shocks, and the unemployed trade in the spot market.

The role of duality in linear programming is exploited to describe decentralization and the welfare theorems. Once a planner's problem is set without consideration of a decentralized economy, economic interpretation of dual linear program characterizes an environment (markets, public randomization devices, contract arbitrageurs, and insurers) of efficiently decentralized economy.

*I am grateful for the guidance of Joseph M. Ostroy. I thank William R. Zame for extended discussion and helpful comments. I also thank Sushil Bikhchandani, Harold L. Cole, Bryan Ellickson, Ichiro Obara, David Rahman, and seminar participants at the UCLA theory proseminar for their helpful comments. The most recent version of this paper is at http://joonsong.bol.ucla.edu/. E-mail: joonsong@ucla.edu

Contents

1 Introduction

Motivation

Information asymmetry inside an organization is an important issue in economics. The partial equilibrium analysis approach (see Grossman and Hart (1983)) has been employed to characterize the optimal wage scheme from the principal's point of view. It summarizes the effect of competition by a single parameter of the agent's outside option. The general equilibrium approach following Prescott and Townsend (1984) has been employed to characterize the incentive-constrained efficient allocation, in which principals are perfectly competitive that they get zero profit. The incompleteness of these two approaches is (i) that competition is not fully captured, (ii) that bargaining power is given to principals or agents by assumption (for example, Bennardo and Chiappori (2001)), and (iii) that the team formation problem is largely ignored. In order to determine bargaining power, study of matching or team formation is unavoidable.

The team formation (Cole and Prescott (1997), Ellickson, Grodal, Scotchmer, and Zame (1997, 2001), Makkowski and Ostroy (2003)) and matching literatures have not considered moral hazard problem within teams until recently. Rahman (2005a) and Zame (2005) study team formation models with incentive problems. Similarities and differences of the two to this paper are mentioned later.

Objective

This paper considers *incentive-constrained* efficient contractual matching of a finite number of individuals and continuums of individuals in the presence of moral hazard. A finite number of individuals form a team/firm in order to produce stochastic output. Moral hazard problems exist within firms since the efforts that individuals make are not verifiable.

The goals of the paper are (i) to characterize conditions for incentive-constrained efficiency and (ii) to show a convergence of the finite model to the continuum one. A few characteristics of the finite model are eliminated in the continuum; therefore the convergence shows what are implicitly assumed in the continuum model. Linear programming is used as a methodology. Once a planners problem is set as a linear programming problem, economic interpretation of duality in linear programming characterizes an idealized market environment (institutions, technologies such as public randomization, and markets) for achieving efficiency.

Models and Results

In the finite economy, a market of lotteries on matchings with Lindahl prices, public randomization devices, contract arbitrageurs, and insurers are by linear programming found to be necessary for efficiency.

Random matching improves efficiency by creating the uncertainty to relax incentive compatibility constraints more. In order to implement random matching, a lottery market for matching and a public randomization device for the lotteries are required. (See Prescott and Shell (2002)) Unlimitedly supplied contract arbitrageurs in the finite economy are expected money maximizers inventing exclusive random contracts that (i) depend on the entire state of the economy and (ii) prohibit individuals from trading in the spot market.

Since arbitrageurs compete, they will try to create lotteries as favorable as possible to individuals in the sense that the contract insures individuals as much as possible without breaking proper incentive compatibility constraints. Contract arbitrageurs and individuals trade the lotteries. A contract arbitrageur could be active or inactive depending on whether his contract is chosen by the public randomization device or not. Another public randomization device is required to implement random contracts invented by contract arbitrageurs. Even though contract arbitrageurs are risk-neutral, they have to deliver non-money commodities promised by the contracts. Therefore, they purchase insurance for non-money commodities to eliminate the risk; hence, it is shown that access to the insurance market by a firm promotes efficiency, while the access to the market by an individual is not desirable. Lastly required are unlimitedly supplied insurers as expected money maximizers committed to avoid cream-skimming. Markets for lotteries on matchings and insurance take place before the realization of matching. Contract arbitrageurs and insurers end up with zero profit since they are unlimitedly supplied.

The insurer's commitment to avoid cream-skimming means that the insurer cross-subsidizes; i.e., the insurance premium and the value of the expected payments do not coincide. However, contract arbitrageurs get zero expected profit since the subsidies are channeled to individuals through contract arbitrageurs. Therefore, a combined welfare theorem is proved in the finite model, which states that the entire utility frontier could be decentralized; hence, there is no distinction between the first and the second welfare theorems. Mathematically, I illustrate that the combined welfare theorems is from two characteristics of the model: the matching structure and/or firms are essentially public goods, and pricing in the model is non-linear. The uncertainty structure in the finite economy is endogenously determined by the matching structure of the economy; hence, my choice directly effects others' decisions. The price of a lottery on a contract specifying effort and consumption cannot be linearly decomposed into prices of consumption and effort. These public good characteristics of matching and the non-linear pricing yields a large degree of freedom in choosing prices; therefore the entire utility frontier can be decentralized in the finite model.

In an economy with continuums of individuals in which contracts are functions of idiosyncratic shocks, the requirement found by linear programming for efficiency are the following: a market of lotteries on matchings with Lindahl prices, common randomization devices, and contract arbitrageurs.

The role of the public randomization device implementing random contracts and the restrictions of insurers disappear as the finite economy converges to the continuum one. In the finite economy, the randomization device implementing random contracts needs to accommodate and signal enormous amounts of information in the sense that all the firms' random contracts are correlated. However, in the continuum economy, each firm possesses its own randomization device for the exercise of its contract, and the randomization devices are not correlated. In other words, the correlation amongst the contracts across firms can become independent as the size of the economy grows. In the elimination of the public randomization device, elimination of correlation among contracts is obtained by increasing the randomness within contracts, therefore it is shown that random contracts are not only for the relaxation of incentive compatibility constraints, but also for simplicity of contracts in the sense that information outside firms is unnecessary to implement contracts. Uhlig (1996) raises the following question: what could be a limit of a finite economy where contracts depend upon shocks of all individuals in the economy. This paper provides an answer to the question for an economy. This finding hints a justification of the contracts depending only on idiosyncratic shocks in other general equilibrium contract models as well.

Insurers in the finite economy are committed to taking care of all the risk involved in the economy. Without commitment, they can sell more insurance to some firms if the insurance premium exceeds the expected payment, and none to others. In other words, insurers are committed not to engage in cream-skimming; hence they cross-subsidize across firms. Insurers committed to pooling the risks in this way are hardly observed. Also, insurers need to know all the details of the economy to calculate the right prices in order to bid for the license, which does not seem realistic. The role of the insurers in the continuum economy is replaced by an ex-post market that is a mixture of spot market trading by the unemployed and contingent claims market trading by contract arbitrageurs. Therefore, there is no need for insurers to achieve efficiency in the continuum economy.

The combined welfare theorem does not hold in the continuum economy. Given that the continuum economy is a limit of the finite economy, having two different welfare theorems in the continuum and the finite models seem contradictory to the convergence. However, since the insurer in the finite model is to take care of all the risk in the economy, the risk premium does not necessarily have to reflect the true value of the insurance, i.e. the expected payment from the insurance. If the insurer were to engage in cream-skimming, the insurer would not sell insurance to some of firms. In other words, the inequality of the value and the price of insurance means that the insurer cross-subsidizes firms. On the other hand, there is no cross-subsidy in the continuum model since the ex-post market replaces the role of insurer. Therefore the two different welfare theorems are in fact not contradictory.

The results of (1) elimination of the public randomization device implementing random contracts, (2) replacement of insurers with the ex-post market, (3) no cross-subsidies for efficiency in the continuum economy, and (4) the combined welfare theorems in the finite economy are from the formulation of resource constraints in linear programming.

I show that trades of lotteries are necessary for efficiency even in the continuum economy. Therefore, it is shown that efficiency typically requires the following: (i) individuals of the same type to obtain different expected utilities when assigned to different firms, (ii) compensating wage differentials which equalize the utilities of individuals in different firms are generally incompatible with efficiency, and (iii) competitive equilibria are efficient if the lotteries market exists. A continuum of each type possess their own common randomization device that direct team assignment. Common randomization devices are independent across types. Bennardo and Piccolo (2005) study a general equilibrium model where agents' preferences, productivity, and labor endowments depend on their health status, and they found the similar results. Discussion is detailed in section 3.5.

Random matching is due to the nature of indivisibility of teams coupled with the assumption of non-transferable utilities. Random contract is due to the nature of the incentive compatibility constraints. Indivisibility of teams and the incentive compatibility constraints exist even in the continuum model; therefore random matching and random contract exist even in the continuum model.

The condition of no spot market trading by the employed is derived by observing the linearity and non-linearity of prices. Even though the price of lottery is linear in the space of probabilities as in Cole and Prescott (1997), the price is not linear in the space of commodities in the sense that the price of the lottery cannot be linearly decomposed into commodity prices. Therefore, arbitrage opportunity exists. Exclusiveness of contracts is to avoid arbitrage; hence, it is to achieve efficiency. Requirement of non-linear prices in the presence of non-convex domain is observed in many other literatures. For example, see Bisin and Gottardi (1999) and Hellwig (2005). Since the non-convexity due to indivisibility nature of teams and the incentive compatibility exists even in the continuum model, the condition of no spot market trading by the employed does not vanish.

The technical contribution is to formulate a team model with finite number of individuals using linear programming. This paper is an extensive exercise of the application of linear programming to general equilibrium theory. The roles of definition of equilibria and welfare theorems have been implicitly reversed in the general equilibrium literature: welfare theorems are proved after the definition of equilibria, but the definition was properly stated in order to satisfy the theorems. The approach here is to explicate the reversed roles of the definition and the theorems. Once the planner's problem is set as a linear programming problem, the duality of linear programming gives a direct interpretation of the welfare theorems, and the definition of equilibria is obtained by investigating the dual linear programming problem. The methodology here can be used as a general procedure for characterizing an efficient decentralized economy.

Literature Review

The related team literatures can be classified into four categories.

Firstly, there are literatures considering a model with continuums of individuals where lotteries are used to improve efficiency of economy. Cole and Prescott (1997) consider a team model without informational asymmetry. Since preferences over risk are exploited for efficiency, lottery market is introduced. Prescott and Townsend (2000) deal with the problem of hiring monitors to relax incentive constraints for workers in general equilibrium. However, by assuming homogeneous individuals, the complementarity and substitutability of individuals within teams are ignored.

Secondly, there are literatures considering models with continuums of individuals where lotteries are not used. For decentralization without lotteries, same type individuals get the same utility even when their choices are different. The difference of Cole and Prescott (1997) and literatures in this category is further discussed in section 3.5. Ellickson, Grodal, Scotchmer, and Zame (1999) consider a team model without informational asymmetry. Their model is more general than this paper in the sense that individuals can join many teams with the presence of externality inside teams. Makowski and Ostroy (2003) is the first paper to use the duality of linear programming to model a team economy. Rahman (2005a) extends the notion of contractual pricing of Makowski and Ostroy (2003), proposes a version of price taking equilibrium to formalized the idea that individuals compete to play games, and shows that anonymous pricing of jobs fail to decentralized incentive-constrained efficiency. With quasi-linearity of utilities assumed in Makowski and Ostroy (2003) and Rahman (2005a), a market for lotteries on matching is not required for decentralization since money can compensate the difference of utilities from non-money commodities when same type individuals are assigned to different occupations. Zame (2005) considers a team model with moral hazard and adverse selection. He sets up a decentralized economy, and shows that the equilibria may not be Pareto optimal. Moreover, equilibria may even be Pareto ranked. His paper focuses on a proper concept of equilibrium refinement, *Population Perfect equilibrium*. My paper's approach is to set up a planner's problem, and to investigate the decentralization possibility; hence, the welfare theorems are derived automatically, and refinement is unnecessary. In short, Zame (2005) characterizes equilibria under a given environment, while I derive necessary conditions on an environment for it to be incentive-cocnstrained efficient.

Thirdly, even though Garatt (1995) is not a team model paper, he investigates the consequence of indivisibility in a model with finite number of individuals. He finds that a public randomization device (sunspot device) is required for efficiency. My paper shares a similar phenomenon in that a public randomization device is required to implement random contracts correlated across teams in the finite model.

Lastly, Ellickson, Grodal, Scotchmer, and Zame (2001) is a finite analogue of their previous paper. They show that an approximate core can be approximately decentralized by prices for private goods and for club memberships.

Related linear programming literatures include the followings. Makowski and Ostory (1996) is the first paper utilizing the duality of linear programming to characterize decentralization. Makowski and Ostroy (2003) and Rahman (2005a) extend the methodology to model team economies. Jerez (2003) identifies welfare effects associated with the incentives of the agents to truthfully reveal their private information. By linear programming, Rahman (2005b) explores the role of monitoring within a team in a general equilibrium setting with quasi-linearity of utilities.

Ostroy and Song (2005) explore a new perspective on correlated equilibrium of games as public goods decision. The finite model of this paper can be understood as an extended version of correlated equilibrium (see Myerson (1991)) in the sense that each team does not observe how other individuals are matched. All the utility frontier can be decentralized in Ostroy and Song (2005) as well.

Other related literatures include Dam and Pèrez-Castrillo (2001), Serfes (2003), and Magill and Quinzii (2005) that are interested in characterizing more specific contractual forms in a more simplified environment.

Outline

In section 2, I present a model of the finite economy with bilateral matching, and characterize an environment of decentralized economy. Bilateral matching is considered only for the simplicity of notation and exposition. The generalization to teams with arbitrary number of members follows in section 5. Section 3 considers continuum economy with bilateral matching. Generalization of the the economies to team economies is straightforward, and is provided in section 5. Section 4 shows convergence of the finite model to the continuum one. Section 6 discusses more generalizations: inclusion of economy-wide global shocks and play of correlated games inside teams. Section 7 is summary and conclusion. All the proofs are in appendices, unless otherwise mentioned.

2 A Matching Problem with Moral Hazard: Finite Economy

The linear programming formulation of the planner's problem without considering decentralization is proposed in section 2.1. In section 2.2, the market environment required for the decentralization is naturally derived by the dual linear programming problem. The proof of the welfare theorem is in Section 2.3. Discussion on the welfare theorems is in section 2.4.

2.1 Planner's problem

The planner's problem is formulated as a linear programming problem.

Assignments of Matching and Efforts: There are two heterogeneous populations, I and J. A typical individual in I, J, or $I \cup J$ is denoted by i, j, or k. Each individual can work only when each is matched with one from the other population. Let $\mathcal{E}$ be a finite set of efforts. When i and j are matched, it is said that *firm* (ij) *is formed*. When they are matched, they contract upon their efforts, which are denoted by $(e_i, e_j) \in \mathcal{E}^2$. Although I assume that the support of efforts is the same for all the individuals, it is without loss of generality since it can be relaxed by assuming infinite costs of certain efforts for certain individuals. Generalization to the team model is straightforward and discussed in section 5.

Contractual (extended) Matching Function: Contractual matching function is

$$\begin{aligned} A \quad &: \quad I \cup J \rightarrow \left[(I \cup J) \times \mathcal{E}^2\right] \cup \{(\emptyset, \emptyset, \emptyset)\} \\ &s.t. \quad A(i) = (j, e_i, e_j) \text{ iff } A(j) = (i, e_i, e_j) \end{aligned}$$

If i is not matched to anybody, $A(i) = (\emptyset, \emptyset, \emptyset)$. For the simplicity of notation, I write $(i, j, e_i, e_j) \in A$ if $A(i) = (j, e_i, e_j)$. I also write $(i, j) \in A$ if i and j are matched with certain efforts, and $(i, \emptyset) \in A$ if $A(i) = (\emptyset, \emptyset, \emptyset)$. Define [1] $\overline{I} := I \cup \{\emptyset\}$, $\overline{J} := J \cup \{\emptyset\}$, and $\overline{\mathcal{E}} := \mathcal{E} \cup \{\emptyset\}$. Define $\mathcal{A}$ to be the set of all contractual functions.

Note that the contractual matching function assigns not only pairs but also the effort for each assigned pair.

Technology, state, and matching: The state of firm (ij) is denoted by $s_{ij} \in S = \{0, 1, \ldots, S\}$. For a given matching function A, the state of the economy is $\mathbf{s}_A = (s_{ij})_{(i,j) \in A}$. $\mathbf{s}_A$ is realized with probability $\Pr(\mathbf{s}; A)$. In other words, the realization of state $\mathbf{s}$ is effected by the efforts specified in A. (Note that, if shocks for firms are independent, $\Pr(\mathbf{s}; A)$ is the product of the probability of individual shocks. For example, $\Pr(\mathbf{s}; A) = \prod_{(i,j,e_i,e_j) \in A} \varphi(s_{ij}; e_i, e_j)$ where $\varphi(s_{ij}; e_i, e_j)$ is the independent probability of s_{ij} in firm (ij) implementing efforts (e_i, e_j).)

The output of individual firm (ij) at state s_{ij} is $q(s_{ij})$. There is a one-to-one relationship between s and q, however, I keep this notation to reduce any possible confusion. I often write $q_{ij}(\mathbf{s}) := q(s_{ij})$ when s_{ij} is an element in $\mathbf{s}$. Also, $q(0) = 0$, and $\varphi(0_{i\emptyset}; \emptyset, \emptyset) = 1$.

Commodities, Allocation, and Utility Function: There are L non-money commodities and one money commodity. Let $\Delta z_i^{\mathbf{s}}$ be a randomized assignment of consumption at state $\mathbf{s}$. Let $\Delta z_i := (\Delta z_i^{\mathbf{s}})_{\mathbf{s} \in \mathbf{S}}$ be a randomized assignment of consumption defined for all states.

Definition 1 (Assignment/Allocation) *Assignment/Allocation* $(A, (\Delta z_k))$ *specifies* [**Assignment**] *how individuals are matched with which efforts, A, and* [**Allocation**] *what the randomized consumption at the realization of the state of economy* $(A, \mathbf{s})$ *are.*

The expected utility of i for given $(A, (z_k))$ is denoted by $\sum_{\mathbf{s} \in \mathbf{S}_A} v_i(z_i^{\mathbf{s}}) \Pr(\mathbf{s}; A) - E_i(e_i)$ where $E_i(e_i)$ represents the utility cost of effort e_i. $v_i(\cdot)$ is strictly increasing, strictly concave, and differentiable. The utility function of dummy individual $\emptyset$, the utility function is defined as $v_\emptyset(z) = 0$ if $z = 0$, $-\infty$ otherwise.

Allocations as functions of the state of the economy: Allocations are dependent upon the uncertainty of the economy.

Notations of a few probabilities

$X(A)$**:** probability that A is realized

[1] These definitions are for dummy individuals, who are 'matched' to the unmatched. Under the assumption that everybody is matched, all the notation for dummy individuals are useless. Readers who are not interested in formality can ignore the difference between I and $\overline{I}$, and J and $\overline{J}$, ignore all the description about dummy individuals, and assume that everybody is matched.

$X(A, (z_k)_{k \in \overline{I} \cup \overline{J}})$**:** probability that A is realized, and non-random consumption z_k are awarded to individuals

$X_{ij}(A, z_i, z_j)$**:** probability that A is realized with i and j matched, and i and j consume z_i and z_j

$X_i(A, z_i)$**:** probability that A is realized, and i consumes z_i

$X_i(z_i | A, \Delta z_i)$**:** probability of z_i conditional on A and random allocation Δz_i, $X_i(A, z_i)/\sum_{\tilde{z}_i} X_i(A, \tilde{z}_i)$

$X_i(A, z_i |, j, e_i, e_j, \Delta^{(i,j,e_i,e_j)} z_i)$**:** probability of (A, z_i) conditional on matched team (i, j, e_i, e_j) and random consumption $\Delta^{(i,j,e_i,e_j)} z_i := (\Delta z_i)_{A \ni (i,j,e_i,e_j)}$, $X_i(A, z_i)/\sum_{A \ni (i,j,e_i,e_j)} \sum_{z_i} X_i(A, z_i)$

For individual i, random matching with efforts and consumption is described by probability $X_i(A, z_i)$. For example, if $X_i(A, z_i) = X_i(A, z_i') = X_i(A', z_i'') = 1/3$, individual i is situated in matching structure A with probability 2/3, and in A' with probability 1/3. In the case that A is realized i consumes z_i or z_i' with probability 1/2. In the case that individual i is situated in A', individual i consumes z_i'' with certainty.

$X_i(A, z_i)$ is a marginal probability of $X(A, (z_k))$ by $X_i(A, z_i) = \sum_{z_{-i}} X(A, (z_k))$. Technically speaking, $X(A)$ is a probability mass function since $\mathcal{A}$ is a finite set. $X_i(A, z_i)$, $X_{ij}(A, z_i, z_j)$, and $X(A, (z_k))$ are probability density functions since $\mathcal{A} \times \mathbb{R}_+^{L \times |\mathbf{S}|}$, $\mathcal{A} \times \mathbb{R}_+^{2L \times |\mathbf{S}|}$, and $\mathcal{A} \times \mathbb{R}_+^{|I \cup J| \times L \times |\mathbf{S}|}$ are continuum sets. However, they are considered as probability mass functions as if there is only finite support. Moreover, they actually have finite support by Carathéodory's Theorem on Convex Hull. Technical details will be dealt with in appendices.

In order to save space, when there is no confusion, $\Delta^{(i,j,e_i,e_j)} z_i$ is often written as Δz_i. For example, $(i, j, e_i, e_j, \Delta z_i, \Delta z_j)$ should mean $(i, j, e_i, e_j, \Delta^{(i,j,e_i,e_j)} z_i, \Delta^{(i,j,e_i,e_j)} z_j)$ since the description does not include A.

Choice of Information Structure: Suppose matching with allocation with Δz_i and Δz_j, $(i, j, e_i, e_j, \Delta z_i, \Delta z_j)$, was realized. After i and j are matched, they know that they are matched. The following assumptions can be employed.

Assumption 1 (*Local* Information Structure) *Once i and j are matched with (e_i, e_j), they do NOT observe how others were matched.*

Alternatively,

Assumption 2 (*Global* Information Structure) *Once i and j are matched with (e_i, e_j), they DO observe how others were matched.*

There could be many possible information structure in between, but I only concentrate on these two cases.

Allowing random contracts, the *local incentive compatibility constraint* (IC_l) is

$$\sum_{A \ni (i,j,e_i,e_j)} \sum_{z_i} DG_i(e_i' | A, z_i) X_{ij}(A, z_i, z_j) \leq 0 \qquad [IC_l]$$

where deviation gain $DG_i(e_i'|A, z_i)$ is defined by

$$DG_i(e_i'|A, z_i) := \left[\sum_{\mathbf{s}\in\mathbf{S}_A} v_i(z_i^{\mathbf{s}})\Pr(\mathbf{s}; A|e_i') - E_i(e_i')\right] - \left[\sum_{\mathbf{s}\in\mathbf{S}_A} v_i(z_i^{\mathbf{s}})\Pr(\mathbf{s}; A) - E_i(e_i)\right]$$

where $\Pr(\mathbf{s}; A|e_i')$ is same as $\Pr(\mathbf{s}; A)$ with the only difference being $A(i) = (j, e_i', e_j)$ (or alternatively $A(j) = (i, e_i', e_j)$). Since they do not observe the matching of others, uncertainties are two folds, A and $\mathbf{s} \in \mathbf{S}_A$.

Under the *global* information structure, the *global incentive compatibility constraint* (IC_g) is

$$\sum_{z_i} DG_i(e_i'|A, z_i)X_{ij}(A, z_i, z_j) \leq 0 \qquad [IC_g]$$

Note that I used unconditional probability $X_{ij}(A, z_i, z_j)$ so that the constraints are linear in terms of $X_{ij}(\cdot, \cdot, \cdot)$, since I want to model the planner's problem as a linear programming problem.

Although I assumed that there is no Assignment/Allocation randomizing the efforts, my model is general enough to incorporate random contracts upon efforts. The extension of contracts randomizing effort is illustrated in Rahman (2005a). The extension of my model incorporating random contracts upon efforts is straightforward and mentioned in section 6.3.

2.1.1 Linear Programming Formulation of Planner's Problem

I formulate the objective function of the planner as a linear function, and all the constraints of the planner as linear constraints.

Objective Function: The following is the expected utility for individual k.

$$U_k(X_k(\cdot, \cdot)) := \sum_A \sum_{z_k} \left[\sum_{\mathbf{s}\in\mathbf{S}_A} v_k(z_k^{\mathbf{s}})\Pr(\mathbf{s}; A) - E_k(e_k)\right] X_k(A, z_k)$$

since $X_k(A, z_k)$ is the probability that the planner puts individual k in matching A with consumption z_k. Again, in principle, I need to have $\int_{z_k}$ instead of $\sum_{z_k}$, however, I keep the notation for simplicity. All the technical details are delegated to appendices.

As each individual's weight changes in planner's objective, all the Pareto efficient allocation can be obtained. The objective function for the planner is

$$\sum_i \lambda_i U_i(X_i(\cdot, \cdot)) + \sum_j \lambda_j U_j(X_j(\cdot, \cdot))$$

where $\lambda := (\lambda_i)_{i\in I\cup J}$ is a weight profile such that $\lambda \gg 0$.

Probability constraints: Sum of probabilities $X_k(A, z_k)$ has to be equal to 1

$$\sum_A \sum_{z_k} X_k(A, z_k) = 1, \forall k \in I \cup J$$

Matching constraints: $X_i(A, z_i)$ cannot be arbitrary. Let $X_{ij}(A, z_i, z_j)$ be the probability that i and j are matched and situated in A with consumption z_i and z_j. Then the following must hold.

$$\begin{aligned} X_{ij}(A, z_i, z_k) &= \sum_{z_{-i}, z_{-j}} X(A, (z_k)_{k \in I \cup J}) \text{ if } (i,j) \in A \\ X_i(A, z_i) &= \sum_{z_j} X_{ij}(A, z_i, z_j), \forall A, i \in I, j \in \overline{J}, z_i \text{ where } (i,j) \in A \\ X_j(A, z_j) &= \sum_{z_i} X_{ij}(A, z_i, z_j), \forall A, j \in J, i \in \overline{I}, z_j \text{ where } (i,j) \in A \end{aligned}$$

In other words, $X_i(A, z_i)$ is the marginal probability of $X(A, (z_k))$.

Resource Constraint: For any realized matching A and state $\mathbf{s}_A$, the consumption cannot be larger than what was produced in the economy. Therefore,

$$\sum_k z_k^{\mathbf{s}_A} \leq \sum_{(i,j) \in A} q(s_{ij}), \forall \mathbf{s}_A \in \mathbf{S}_A, A$$

The constraint can be written as

$$\left[\sum_k z_k^{\mathbf{s}_A} - \sum_{(i,j) \in A} q_{ij}(\mathbf{s}) \right] X(A, (z_k)) \leq 0, \forall \mathbf{s}_A \in \mathbf{S}_A, A, (z_k)$$

Incentive Compatibility Constraints: I consider three cases: (1) no incentive problem, (2) global incentive compatibility constraint, and (3) local incentive compatibility constraint. It is trivial that IC_l and IC_g are linear constraints.

In summary, the planner's problem with weight profile $(\lambda_k)_{k \in I \cup J}$ is

$$\begin{aligned} (P) \quad \max \quad & \sum_k \sum_{z_k} \sum_A \lambda_k \left[\sum_{\mathbf{s} \in \mathbf{S}_A} v_k(z_k^{\mathbf{s}}) \Pr(\mathbf{s}; A) - E_k(e_k) \right] X_k(A, z_k) \\ s.t. \quad & \sum_A \sum_{z_k} X_k(A, z_k) = 1, \forall k \in I \cup J \\ & X_i(A, z_i) - \sum_{z_j} X_{ij}(A, z_i, z_j) = 0, X_j(A, z_j) - \sum_{z_i} X_{ij}(A, z_i, z_j) = 0, \forall j, A, z_i, z_j \\ & X_{ij}(A, z_i, z_j) - \sum_{z_{-i}, z_{-j}} X(A, (z_k)) = 0, \forall i, j, z_i, z_j, A \\ & \left[\sum_k z_k^{\mathbf{s}_A} - \sum_{(i,j) \in A} q_{ij}(\mathbf{s}) \right] X(A, (z_k)) \leq 0, \forall \mathbf{s}_A \in \mathbf{S}_A, A \\ & No\ IC \text{ or } IC_g \text{ or } IC_l \\ & X_k(A, z_k), X_{ij}(A, z_i, z_j), X(A, (z_k)) \geq 0 \end{aligned}$$

Non-Binding Resource constraint: In the economy without moral hazard problem, no resource is wasted by the planner. With incentive compatibility constraints, it is optimal to waste resources at certain states.

For example, suppose there are only two states of the economy, the *lower output state* and a *higher output state*. By wasting resources in the *lower output state*, the planner could implement *higher effort*. If the higher probability of the *higher output state* due to the higher effort level outweighs the waste of the resources, the planner would waste resource of the *lower output state*. Holmstrom (1982) emphasizes the role of the principals as a budget balance breaker. If a resource is wasted in state **s**, the dual variable for the resource at state **s** is zero by complementary slackness of linear programming.

Assumption 3 *The domain of the planner's problem is not empty.*

Proposition 1 *A solution for the planners problem exists.*

Proof. From the assumption on $v_i(\cdot)$, the non-emptiness of the domain, and Carathéodory's Theorem on convex hull, a maximum exists. ∎

Note that the solution of the planner's problem is incentive-constrained efficient by definition.

Definition 2 *A random assignment of firms and consumption is* *incentive-constrained efficient* *if the allocation solves the planner's problem.*

The planner's problem can be considered as a linear programming problem with an infinite number of variables. It also has an infinite number of constraints since the second constraint has to be satisfied for all z_i and z_j. However, in the appendices, I show that it can be considered as if it is a linear programming problem of finite variables and finite constraints, since the support of $X(A,(z_k))$ is finite by Carathéodory's Theorem on Convex Hull.

If IC_g is assumed, there exists an integral solution for the matching assignment. In other words, the support of $\sum_{(z_k)} X(A,(z_k))$ is a singleton. However, the consumption allocation can be random. In other words, for a given A, $X_i(A,\cdot)$ is a non-degenerate random variable. Moreover, the random assignments of consumption for each individual are correlated, since $X_i(A,z_i)$ is a marginal probability of $X_{ij}(A,z_i,z_j)$, which is the marginal probability of $X(A,(z_k))$ by the primal constraints. The benefit of the using random contracts has been pointed by many scholars (see Prescott and Townsend (1984) and Arnott and Stiglitz (1988)).

On the other hand, if IC_l is assumed, there does not exist an integral solution for the matching in general. By creating uncertainty on matching A, it is possible to relax the incentive compatibility constraints further. Again, it is well known that the uncertainty makes it easy to implement higher effort with lower cost. The reason for the usage of random matching under IC_l, but not under IC_g, is because of the assumption of the information structure. Under IC_l, the effort is made before A is fully known, while under IC_g the effort is made after matching A is known. Apparently, there is no role for relaxing incentive compatibility constraints by random matching under IC_g.

Proposition 2 *(0) Without the incentive compatibility constraints, there exists a solution of non-random matching and non-random assignment of consumption.*

(1) Under IC_g, random assignment of consumption is used in general, but there exists a non-random assignment of matching A.

(2) Under IC_l, random assignment of consumption is used in general, and also non-degenrate random assignment of matching A is used in general.

(3) Under both of IC_l and IC_g, the random assignment of consumption for each individual are correlated.

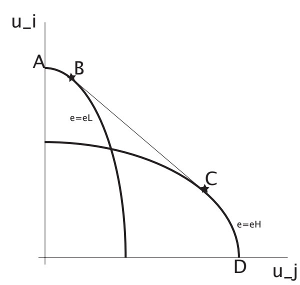
Proof. (0) From the concave utility functions, the result follows. (1) By backward induction. (2, 3) By relaxation of IC by creating uncertainties. ∎

Note that the existence of the solution of the non-random matching assignment in the economy without moral hazard or with IC_g does not necessarily imply that there does not exist a random matching assignment.

Example 1 *The economy has only two economic agents, i and j. Only i can choose efforts, $e \in \{e_H, e_L\}$. In the figure below, the two elliptical thick lines are utility frontiers that depend on the effort of individual i. Let the slope of line $\overline{BC}$ be $\ell := \lambda_j/\lambda_i$. For weights other than ℓ, the planer's solution does not contain random matching. (Note that matching specifies not only the matching of the individuals, but also efforts.)*

For weight ℓ, even though there exist two non-random assignments of matching (B and C), the planner also can choose any point between them, which is a random assignment.

In other words, the non-random assignment of A in the economy without moral hazard or with IC_g restricts the attainable utility frontier.



2.2 Decentralization of Efficient Assignment

The dual linear programming problem of the planner's program specifies a market environment where the decentralized economy replicates the optimization of the planner's. The market environment includes commodities, prices of the commodities, timing of relevant markets for the commodities, necessary commitment, technologies such as public randomization devices, and arbitrageurs of the commodities. Detailed derivation of the market environment for the decentralization is provided in section 2.3, and I state market environment here without derivation.

Commodities: In an economy without moral hazard or with IC_g, commodities are lotteries on matching structure and random consumption, $(A, \Delta z_i)$. In the economy with IC_l, commodities are lotteries on contract $(i, j, e_i, e_j, \Delta z_i)$ that specifies who are matched, what random payoffs are, and what the efforts are.

Prices of the Commodities: Prices of the contract is individual-specific, i.e. Lindahl prices. In the economy without moral hazard or with IC_g, price of $(A, \Delta z_i)$ for individual i is denoted by $t_i(A, \Delta z_i)$. In the economy with IC_l, price of $(i, j, e_i, e_j, \Delta z_i)$ is denoted by $t_i(j, e_i, e_j, \Delta z_i)$.

Players of the economy: Besides the individuals in the planner's problem, unlimitedly supplied competitive contract arbitrageurs and insurers are derived from the dual linear programming.

Contract arbitrageurs are expected money maximizers, and specialize in contract writing. They try to maximize expected money income by selling lotteries on matching A to individuals, and by innovating a contract. Although they are risk neutral in terms of money, they get negative infinite payoff if they do not fulfill the contract, i.e. if they fail to deliver the non-money good promised in the contract to the members of the firm. Therefore, they have to insure themselves through an insurer. Since their supplies are unlimited, they end up with zero profit in equilibrium. Since they compete, contracts will insure individuals as much as possible with the restriction of incentive compatibility.

Insurers are also expected money maximizers and specialize in insuring all the risks in the economy. In order to get a license to do business from the planner, the insurers must be committed to taking care of all the risks faced by the firms in the economy. In other words, the insurers must meet all the demand from all the firms. For example, if the economy has two firms, T and T', the insurer must sell the amount of insurance that each firm wants to buy even if the insurer's profit from selling to T' is negative. However, as long as the combined profit from T and T' is non-negative, the insurer is willing to be committed to selling insurance to the economy. In a competitive environment, the equilibrium expected payoff to the insurers would be zero.

Markets and Timing of Markets: There are a market for lotteries on contracts and a market for insurance. The timing is described below.

The 0th stage (*ex-ante* phase): Individuals and contract arbitrageurs trade lotteries on contracts: (A, z_i) in the economy without moral hazard, $(A, \Delta z_i)$ in *global information structure*, and $(i, j, e_i, e_j, \Delta z_i)$ in *local information structure*. Contract arbitrageurs and insurers trade insurance for non-money commodities.

The 1st stage (*interim* phase): Matching A is realized, and is known to individuals under *global information structure*, but not in *local information structure*.

The 2nd stage: i and j choose efforts.

The 3rd stage: The matching and state of the economy are revealed. The contracts are exercised and they consume.

Note that the price of a contract lottery can be negative, in which case 'purchase' means 'sale', and 'sale' means 'purchase'.

In the 3rd stage, agents are committed to the consumption specified by the contracts. Therefore they do not trade after the exercise of contracts. Section 6.4 considers the case where individuals do not have the technology of commitment to a certain consumption bundle (i.e. they trade in a spot market after the realization of all uncertainty).

Commitment: The aforementioned timing gives information about what kind of commitments are required in the economy. Firstly, individuals are committed to the consumption specified by the contracts. It is known that *ex-ante* efficiency can be achieved by implementing *ex-post* inefficiency in the presence of moral hazard. The planner can achieve *ex-post* inefficiency in certain states by not equalizing marginal rate of substitution. If individuals were allowed to trade in a spot market (the 3rd stage), they can obtain higher utility. However, that possibility would break the incentive compatibility constraint.

It is also assumed that each individual has no private access to a contingent market. With the private access to a contingent market, an economic agent will try to smooth consumption over states; hence, the contracted effort cannot be enforced (see Tommasi and Weinschelbaum (2004)).

Existence of public randomization device: In order to exercise random contracts, a public randomization device is required. For a realized uncertainty $\mathbf{s}_A$, firms pay a randomized payoff $z_i^{\mathbf{s}}$ with probability $X_i(A, z_i)/\sum_{\tilde{z}_i} X_i(A, \tilde{z}_i)$. However, all the firms' exercise of random contracts have to be correlated by the primal constraints

$$X_{ij}(A, z_i, z_j) - \sum_{z_{-i}, z_{-j}} X(A, (z_k)) = 0, X_i(A, z_i) - \sum_{z_j} X_{ij}(A, z_i, z_j) = 0, X_j(A, z_j) - \sum_{z_i} X_{ij}(A, z_i, z_j) = 0$$

in order to replicate the solution of the planner's problem.

Definition 3 (Definition of Equilibrium) 1. Individual Optimization: *Individuals buy lotteries on contract (A, z_i) ($(A, \Delta z_i)$ or $(i, j, e_i, e_j, \Delta z_i)$) at per-unit probability price of $t_i(A, z_i)$ ($t_i(A, \Delta z_i)$ or $t_i(j, e_i, e_j, \Delta z_i)$) in the economy without moral hazard (with IC_g or with IC_l). Once matching A is realized, individuals i and j choose efforts. After the realization of the state of the economy (A, (z_k)), $\mathbf{s}$, state of the randomization* *device) is known, individuals consume. Formally, the individuals' problems are*

$$NoIC \quad : \quad \max_{Q_i(A,z_i)} \sum_A \sum_{z_i} \left[\sum_{\mathbf{s}} v_i(z_i^{\mathbf{s}}) \Pr(\mathbf{s}; A) - E_i(e_i) \right] Q_i(A, z_i) \;\; s.t. \;\; \sum t_i(A, z_i) Q_i(A, z_i) = 0$$

$$IC_g \quad : \quad \max_{e_i'} \max_{Q_i(A,\Delta z_i)} \sum_A \left\{ \sum_{z_i} \left[\sum_{\mathbf{s}} v_i(z_i^{\mathbf{s}}) \Pr(\mathbf{s}; A') - E_i(e_i') \right] X_i(z_i|A) \right\} Q_i(A, \Delta z_i)$$

$$s.t. \sum_{(A,\Delta z_i)} t_i(A, \Delta z_i) Q_i(A, \Delta z_i) = 0$$

$$IC_l \quad : \quad \max_{e_i'} \max_{Q_i(j,e_i,e_j,\Delta z_i)} \sum_{j,e_i,e_j} \left\{ \sum_{A \ni (i,j,e_i,e_j)} \sum_{z_i} \left[\sum_{\mathbf{s}} v_i(z_i^{\mathbf{s}}) \Pr(\mathbf{s}; A') - E_i(e_i') \right] X_i(A, z_i | j, e_i, e_j) \right\} Q_i(j, e_i, e_j, \Delta z_i)$$

$$s.t. \sum_{(j,e_i,e_j,\Delta z_i)} t_i(j, e_i, e_j, \Delta z_i) Q_i(j, e_i, e_j, \Delta z_i) = 0$$

2. Contract-arbitrageur's Optimization: *Unlimitedly supplied contract-arbitrageurs sell lotteries to individuals at the price of $t_i(A, z_i)$, $t_i(A, \Delta z_i)$, or $t_i(j, e_i, e_j, \Delta z_i)$. Also, contract-arbitrageurs insure themselves through an insurer. Once a contract arbitrageur's contract is picked by the randomization device, she takes charge of the firm. Formally, contract arbitrageurs' problems are*

$$NoIC \quad : \quad \max_{Q_{ij}(A,z_i,z_j)} \sum_{A \ni (i,j,e_i,e_j)} \sum_{z_i,z_j} [t_i(A, z_i) + t_i(A, z_j) - T_{ij}(A, z_i, z_j)] \, Q_{ij}(A, z_i, z_j)$$

$$IC_g \quad : \quad \max_{Q_{ij}(A,\Delta z_i,\Delta z_j)} \sum_{A \ni (i,j,e_i,e_j)} \sum_{\Delta z_i,\Delta z_j} [t_i(A, \Delta z_i) + t_i(A, \Delta z_j) - T_{ij}(A, \Delta z_i, \Delta z_j)] \, Q_{ij}(A, \Delta z_i, \Delta z_j)$$

$$s.t. \; (A, \Delta z_i, \Delta z_j) \textit{ is } \text{globally } \textit{incentive compatible.}$$

$$IC_l \quad : \quad \max_{Q_{ij}(e_i,e_j,\Delta z_i,\Delta z_j)} \sum_{e_i,e_j} \sum_{\Delta z_i,\Delta z_j} [t_i(j, e_i, e_j, \Delta z_i) + t_i(j, e_i, e_j, \Delta z_j) - T_{ij}(e_i, e_j, \Delta z_i, \Delta z_j)] \, Q_{ij}(e_i, e_j, \Delta z_i, \Delta z_j)$$

$$s.t. \; (i, j, e_i, e_j, \Delta z_i) \textit{ is } \text{locally } \textit{incentive compatible.}$$

3. Insurer's Optimization: *Unlimitedly supplied insurers try to maximize the expected payoff, and compete for the license to do business. Formally,*

$$NoIC \quad : \quad \max_{Q(A,(z_k))} \sum_A \sum_{(z_k)} \sum_{(i,j)\in A} T_{ij}(A, z_i, z_j) Q(A, (\Delta z_k)) \;\; s.t. \sum_{(i,j)\in A} [z_i^{\mathbf{s}} + z_j^{\mathbf{s}} - q_{ij}(\mathbf{s})] \le 0$$

$$IC_g \quad : \quad \max_{Q(A,(\Delta z_k))} \sum_A \sum_{(\Delta z_k)} \sum_{(i,j)\in A} \{T_{ij}(A, \Delta z_i, \Delta z_j)\} \, Q(A, (\Delta z_k)) \;\; s.t. \sum_{(i,j)\in A} [z_i^{\mathbf{s}} + z_j^{\mathbf{s}} - q_{ij}(\mathbf{s})] \le 0$$

$$IC_l \quad : \quad \max_{Q(A,(\Delta z_k))} \sum_A \sum_{(\Delta z_k)} \sum_{(i,j)\in A} [T_{ij}(e_i, e_j, \Delta z_i, \Delta z_j)] \, Q(A, (\Delta z_k)) \;\; s.t. \sum_{(i,j)\in A} [z_i^{\mathbf{s}} + z_j^{\mathbf{s}} - q_{ij}(\mathbf{s})] \le 0$$

4. Clearance of Commodity Market Clearance: *The commodity market clears at each $(A, \mathbf{s}_A)$.*

$$\sum_i z_i^{\mathbf{s}} + \sum_j z_j^{\mathbf{s}} \le \sum_{i,j} q_{ij}(\mathbf{s}), \forall \mathbf{s} \in \mathbf{S}_A$$

5. Matching Market Clearance (Consistency): *The matching market clears in the sense that lottery purchases are consistent across the population. Moreover, exercises of contracts have to be coordinated in order to satisfy the resource constraint. In summary,*

$$X_i(A, z_i) = \sum_{z_j} X_{ij}(A, z_i, z_j), \;\; X_{ij}(A, z_i, z_j) = \sum_{z_{-i}, z_{-j}} X(A, (z_k))$$

Theorem 1 (Welfare Theorems) **[The first welfare theorem]** *A price-taking equilibrium with a lottery trade, a randomization device, unlimitedly supplied contract arbitrageurs and insurers is incentive-constrained efficient.* **[The second welfare theorem]** *The planner's assignment/allocation of matching and consumption (constrained by the incentive compatibility constraint) with any weight profile λ can be decentralized by a lottery trade, a randomization device, and competitive contract arbitrageurs and insurers.* **[Characterization]** *Contract arbitrageurs and insurers get zero profit.*

Discussion follows after the proof.

2.3 Characterization of Equilibria and Proof of Theorem 1

Dual linear programming problem of the planner's linear programming problem is derived. From the dual constraints (the constraints of the dual linear programming probelm), a proper definition of price-taking equilibrium is derived.

2.3.1 Dual Linear Programming

Let the dual variables corresponding to each constraint of the planner's linear program be $y_i, y_j, t_i(A, z_i)$, $t_j(A, z_j)$ and $p(A, (z_i), (z_j), \mathbf{s})$. Then the following is the dual linear program without IC constraints.

$$
\begin{aligned}
(D)\min\quad & \sum_i y_i + \sum_j y_j \\
s.t.\quad & y_i \geq \lambda_i \left[\sum_{\mathbf{s}} v_i(z_i^{\mathbf{s}}) \Pr(\mathbf{s}; A) - E_i(e_i) \right] - t_i(A, z_i) \\
& 0 \geq t_i(A, z_i) + t_j(A, z_j) - T_{ij}(A, z_i, z_j) \qquad \text{(under } NoIC\text{)} \\
& 0 \geq \sum_{(i,j,e_i,e_j)\in A} T_{ij}(A, z_i, z_j) + \left[\sum_{(i,j)\in A} q_{ij}(\mathbf{s}) - \sum_i z_i^{\mathbf{s}_A} - \sum_j z_j^{\mathbf{s}_A} \right] p(A, (z_i), (z_j), \mathbf{s}) \\
& p(A, (z_i), (z_j), \mathbf{s}) \geq 0
\end{aligned}
$$

With IC_g, the second dual constraint is

$$
\begin{aligned}
0 \geq\ & t_i(A, z_i) + t_j(A, z_j) - T_{ij}(A, z_i, z_j) \\
& - \sum_{e_i'} \alpha_i(e_i'|A, z_j) DG_i(e_i'|A, z_i) - \sum_{e_j'} \alpha_j(e_j'|A, z_i) DG_i(e_i'|A, z_j) \qquad \text{(under } IC_g\text{)}
\end{aligned}
$$

where $\alpha_i(e_i'|A, z_j)$ is the dual variable for the IC_g constraint.

With IC_l, the second dual constraint is

$$
\begin{aligned}
0 \geq\ & t_i(A, z_i) + t_j(A, z_j) - T_{ij}(A, z_i, z_j) \\
& - \sum_{e_i'} \alpha_i(e_i'|e_i, e_j, z_j) DG_i(e_i'|A, z_i) - \sum_{e_j'} \alpha_j(e_j'|e_i, e_j, z_i) DG_i(e_i'|A, z_j) \qquad \text{(under } IC_l\text{)}
\end{aligned}
$$

where $\alpha_i(e_i'|e_i, e_j, z_i)$ is the dual variable for the IC_l constraint.

Proposition 3 (Fundamental Theorem of Linear Programming) *(1) There exists a solution for each of the three primal linear programs. (2) There exists a solution for each of the three dual linear programs. (3) The values of the primal and dual programming are the same for each of the three programmings.*

Proposition 4 (Complementary Slackness)

$$X_i(A, z_i) > 0 \quad \Rightarrow \quad y_i = \lambda_i \left[\sum_{\mathbf{s}} v_i(z_i^{\mathbf{s}}) \Pr(\mathbf{s}; A) - E_i(e_i) \right] - t_i(A, z_i)$$
$$X_i(A, z_i) = 0 \quad \Leftarrow \quad y_i > \lambda_i \left[\sum_{\mathbf{s}} v_i(z_i^{\mathbf{s}}) \Pr(\mathbf{s}; A) - E_i(e_i) \right] - t_i(A, z_i)$$

Similarly,

$$X_{ij}(A, z_i, z_j) > 0 \quad \Rightarrow \quad \textit{Equality of the relevant second dual constraint}$$
$$X_{ij}(A, z_i, z_j) = 0 \quad \Leftarrow \quad \textit{Strict inequality of the relevant second dual constraint}$$
$$X(A, (z_k)) > 0 \quad \Rightarrow \quad \textit{Equality of the third dual constraint}$$
$$X(A, (z_k)) = 0 \quad \Leftarrow \quad \textit{Strict inequality of the third dual constraint}$$

Before analyzing and interpreting the meaning of the dual constraints, I prove two lemmas that can be shown by a direct application of the fundamental theorem of LP.

Lemma 1 *At an optimal solution of linear programs,*

$$y_k = \lambda_k \sum_A \sum_{z_k} \left[\sum_{\mathbf{s} \in \mathbf{S}_A} v_k(z_k^{\mathbf{s}}) \Pr(\mathbf{s}; A) - E_k(e_k) \right] X_i(A, z_i) - \sum_A \sum_{z_k} t_k(A, z_k) X_i(A, z_i)$$

$$0 = \sum_{z_i} t_i(A, z_i) X_i(A, z_i) + \sum_{z_j} t_j(A, z_j) X_j(A, z_j) - \sum_{z_i, z_j} T_{ij}(A, z_i, z_j) X_{ij}(A, z_i, z_j) \qquad \text{(under } IC_g\text{)}$$

$$0 = \sum_{A \ni (i,j,e_i,e_j)} \sum_{z_i} t_i(A, z_i) X_i(A, z_i) + \sum_{A \ni (i,j,e_i,e_j)} \sum_{z_j} t_j(A, z_j) X_j(A, z_j) - \sum_{A \ni (i,j,e_i,e_j)} \sum_{z_i, z_j} T_{ij}(A, z_i, z_j) X_{ij}(A, z_i, z_j) \qquad \text{(under } IC_l\text{)}$$

$$0 = \sum_A \sum_{(z_k)} T_{ij}(A, z_i, z_j) X(A, (z_k))$$

Lemma 2 *For any weight profile λ, there exists a dual solution, $(y_k, t_k(A, z_k))$ such that*

$$\sum_A \sum_{z_k} t_k(A, z_k) X_i(A, z_i) = 0.$$

2.3.2 Characterization of Decentralization

Historically, the main application of linear programming was revenue maximization with a linear production function for some given inputs. It is well known that the dual linear programming of the revenue maximization problem is 'cost minimization problem'. The 'cost minimization problem' is not cost minimization problem *per se*, since the control variables of the 'cost minimization problem' are prices rather than quantities. However, the optimal prices of the 'cost minimization problem' are not meaningless since they measure the producer's willingness to pay for the (small enough) additional inputs. For a review of linear programming, see appendices.

The planner's problem can be interpreted as a revenue maximization problem. The inputs for the planner's problem are individuals. All other constraints can be interpreted as technological constraints. The dual variables of the individual probability constraint measure the value of the individual, which has the direct interpretation of the individual's utility. The dual value of the second primal constraint measures the value of assignment (A, z_i) to individual i, which is the price of assignment (A, z_i) to individual i. The dual value of the third primal constraint measures the value of assignment (A, z_i, z_j) to firm (ij), which is the price of assignment (A, z_i, z_j) to firm (ij). The dual variable of the fourth primal constraint measures the value of the resource. In the sensitivity analysis of linear programming, the dual variable measures the additional value of the resource. Therefore, the dual variable of the resource constraint is the price of the commodity. Note that the notion of a market has to be introduced in order to define prices. Lastly, the prices of incentives are derived from the last primal constraints, which will be discussed later.

Note that the prices of the second and the third primal constraints measure an assignment to individual i and firm (ij). Since the price is indexed by i and (ij), the natural definition of prices would be Lindahl prices.

2.3.3 Individual Choice: the first dual constraint

The dual variable of the first primal linear program, y_i, is the value of individual i to the planner. The first dual constraint is interpreted as individuals' maximization.

Define prices using the optimal value of the dual linear program.

$$\begin{aligned} t_i(A, \Delta z_i) &:= \sum_{z_i} t_i(A, z_i) Q_i(z_i | A, \Delta z_i) \\ t_i(j, e_i, e_j, \Delta z_i) &:= \sum_{A \ni (i,j,e_i,e_j)} \sum_{z_i} t_i(A, z_i) Q_i(A, z_i | j, e_i, e_j, \Delta z_i) \end{aligned}$$

$t_i(A, \Delta z_i)$ is the price of a random contract that (1) realizes matching A with probability $\frac{\sum_{z_i} Q_i(A, z_i)}{\sum_{\tilde{A}} \sum_{z_i} Q_i(\tilde{A}, z_i)}$, and (2) gives consumption $z_i^{\mathbf{s}}$ at state $(A, \mathbf{s})$ with probability $\sum_{\tilde{z}_i : \tilde{z}_i^{\mathbf{s}} = z_i^{\mathbf{s}}} \frac{Q_i(A, \tilde{z}_i)}{\sum_{\tilde{z}_i} Q_i(A, \tilde{z}_i)}$. $t_i(j, e_i, e_j, \Delta z_i)$ is the price of a random contract that (1) realizes matching (i, j, e_i, e_j) with probability $\sum_{A \ni (i,j,e_i,e_j)} \frac{\sum_{z_i} Q_i(A, z_i)}{\sum_A \sum_{z_i} Q_i(A, z_i)}$, (2) gives consumption $z_i^{\mathbf{s}}$ at state $(A \ni (i, j, e_i, e_j), \mathbf{s})$ with probability $\sum_{\tilde{z}_i^{\mathbf{s}}: \tilde{z}_i = z_i^{\mathbf{s}}} \frac{Q_i(A, \tilde{z}_i)}{\sum_{\tilde{z}_i} Q_i(A, \tilde{z}_i)}$.

By summing up the first dual constraints with arbitrary $Q_i(A, z_i)$, I get

$$y_i \geq \lambda_i \sum_A \sum_{z_i} \left[\sum_{\mathbf{s} \in \mathbf{S}_A} v_i(z_i^{\mathbf{s}}) \Pr(\mathbf{s}; A) - E_i(e_i) \right] Q_i(A, z_i) - \sum_A \sum_{z_i} t_i(A, z_i) Q_i(A, z_i) \qquad \text{[INDV]}$$

$$\Leftrightarrow \; y_i/\lambda_i \geq \sum_A \left\{ \sum_{z_i} \left[\sum_{\mathbf{s} \in \mathbf{S}_A} v_i(z_i^{\mathbf{s}}) \Pr(\mathbf{s}; A) - E_i(e_i) \right] Q_i(z_i | A, \Delta z_i) \right\} Q_i(A, \Delta z_i) - \frac{1}{\lambda_i} \sum_{(A, \Delta z_i)} t_i(A, \Delta z_i) Q_i(A, \Delta z_i)$$

$$\Leftrightarrow \; y_i/\lambda_i \geq \sum_{j, e_i, e_j} \left\{ \sum_{A \ni (i,j,e_i,e_j)} \sum_{z_i} \left[\sum_{\mathbf{s}} v_i(z_i^{\mathbf{s}}) \Pr(\mathbf{s}; A) - E_i(e_i) \right] Q_i(A, z_i | j, e_i, e_j, \Delta z_i) \right\} Q_i(j, e_i, e_j, \Delta z_i)$$
$$- \frac{1}{\lambda_i} \sum_{(j, e_i, e_j, \Delta z_i)} t_i(j, e_i, e_j, \Delta z_i) Q_i(j, e_i, e_j, \Delta z_i)$$

Note that the summation over z_k is meaningless if no random contract is used in the equilibrium. If $Q_i(A, z_i) = X_i(A, z_i)$, then the inequalities are equalities by Complementary Slackness.

From Lemma 2, pick optimal dual variables such that $\sum_A \sum_{z_k} t_k(A, z_k) X_i(A, z_i) = 0$. Then, I have

$$\sum_A t_i(A, \Delta z_i) X_i(A, \Delta z_i) = \sum_{j, e_i, e_j} t_i(j, e_i, e_j, \Delta z_i) X_i(j, e_i, e_j, \Delta z_i) = 0.$$

The meaning of the equality in Lemma 2 is that money expenditure on the lottery purchase is zero if the purchase is the same as that of the planner's solution. On top of that the purchased contracts are incentive compatible by both Complementary Slackness and the primal constraints of the incentive compatibility constraints.

It is shown that non-incentive compatible contracts are never sold by contract arbitrageurs.

Therefore, the above inequality [INDV] summarizes individual i's optimization, since, if individual i has chosen a different probability than that of the planner's, the purchase of the different probability would be infeasible or suboptimal. Also, y_i/λ_i is interpreted as the *ex-ante* utility of i before realization of $(A, \mathbf{s})$. Moreover, $1/\lambda_i$ is i's marginal utility of income. In other words, if individual i were given ϵ amount of money in the beginning, he could have increased his expected utility by changing his purchase of the lotteries.

Remark: By proposition 4, I have

$$\sum_A \sum_{z_i} t_k(A, z_i) X_i(A, z_i) = 0$$

However, $\sum_{A \ni (i,j,e_i,e_j)} \sum_{z_i} t_i(A, z_i) X_i(A, z_i)$ is not zero in the economy with IC_l in general. Therefore, in the economy under IC_l the payoff according to the outcome of the lottery draw is actually different for individuals.

Under the economy with IC_l, after contract $(i, j, e_i, e_j, \Delta z_i, \Delta z_j)$ is known, the realized utility is

$$\frac{y_k}{\lambda_k} + \frac{1}{\lambda_k} \sum_{A \ni (i,j,e_i,e_j)} \sum_{z_k} \sum_{\mathbf{s} \in \mathbf{S}_A} t_k(A, z_k^{\mathbf{s}}) \Pr{}_A^{\mathbf{s}} \frac{X_k(A, z_k)}{\sum_{\tilde{z}_k} X_k(A, \tilde{z}_k)} = \sum_{A \ni (i,j,e_i,e_j)} \sum_{z_k} \left[\sum_{\mathbf{s} \in \mathbf{S}_A} v_k(z_k^{\mathbf{s}}) - E_i(e_i) \right] \frac{X_k(A, z_k)}{\sum_{\tilde{z}_k} X_k(A, \tilde{z}_k)}$$

In general the second term of the left-hand side of the equality is not zero. In other words, IC_l constrained efficiency typically requires agents of the same type to obtain different utility levels when assigned to different team. This result will be elaborated on in the section for the continuum model.

2.3.4 Contract Arbitrageurs' optimization: the second dual constraints

The second dual constraint is the contract arbitrageurs' optimization. There is no probability constraint for contract arbitrageurs, i.e. arbitrageurs are a freely available input to the planner. The price of a freely available input must be zero, so arbitrageurs get zero profit unlike individuals. Being freely available, arbitrageurs are perfectly competitive. They specialize in writing contracts between i and j. The only way for them to gain profit is to innovate a contract between i and j, and to sell a lottery on that contract.

The contract arbitrageur pays non-money commodity z_i and z_j to i and j, which are not necessarily same to $q_{ij}(\mathbf{s})$. Although they are risk neutral in terms of money, they get a payoff of negative infinity if they do not fulfill the contract, i.e. if they fail to deliver the non-money good promised by the contract to the members of the firm. Therefore, they have to insure themselves through an insurer.

In the equilibrium, there are two kinds of arbitrageurs: *ex-ante active* arbitrageurs and *ex-ante inactive* arbitrageurs. *Ex-ante active* arbitrageur sells a positive amount of lotteries on the job assignment, while *ex-ante inactive* arbitrageur sells zero amount of lotteries because her contract is not profitable. Not all ex-ante active arbitrageurs are active *ex-post*. Some are *ex-post active*, and others are *ex-post inactive* depending on the realization of A.

Define the insurance premium using the optimal value of the dual linear programming problem.

$$\begin{aligned}
T_{ij}(A, \Delta z_i, \Delta z_j) &:= \sum_{z_i, z_j} T_{ij}(A, z_i, z_j) Q_{ij}(A, z_i, z_j) \\
T_{ij}(e_i, e_j, \Delta z_i, \Delta z_j) &:= \sum_{A \ni (i,j,e_i,e_j)} \sum_{z_i, z_j} T_{ij}(A, z_i, z_j) Q_{ij}(A, z_i, z_j)
\end{aligned}$$

$T_{ij}(A, \Delta z_i, \Delta z_j)$ is the price of insurance that pays non-money commodity $z_i^{\mathbf{s}} + z_j^{\mathbf{s}} - q_{ij}(\mathbf{s}_A)$ at the realization of $(A \ni (i, j, e_i, e_j), \mathbf{s})$ with probability $\sum_{\tilde{z}_i, \tilde{z}_j : \tilde{z}_i^{\mathbf{s}} = z_i^{\mathbf{s}}, \tilde{z}_j^{\mathbf{s}} = z_j^{\mathbf{s}}} \frac{Q_{ij}(A, \tilde{z}_i, \tilde{z}_j)}{\sum_{\tilde{z}_i, \tilde{z}_j} Q_{ij}(A, \tilde{z}_i, \tilde{z}_j)}$ in the economy with IC_g. In the economy with IC_l $T_{ij}(e_i, e_j, \Delta z_i, \Delta z_j)$ is the price of insurance that pays in a similar way.

The implications of the dual constraints slightly differ across the three economies: without moral hazard problem, with IC_g, and with IC_l. I analyze them one by one.

The economy without moral hazard problem: By summing up the second dual constraints with arbitrary $Q_{ij}(A, z_i, z_j)$, I get

$$0 \geq \sum_{A \ni (i,j,e_i,e_j)} \left[t_i(A, z_i) + t_i(A, z_j) - T_{ij}(A, z_i, z_j) \right] Q_{ij}(A, z_i, z_j) \qquad \text{[ARB\_}NoIC\text{]}$$

If $Q_{ij}(A, z_i, z_j) = X_{ij}(A, z_i, z_j)$, the inequality becomes an equality. Therefore, inequality [ARB_$NoIC$] summarizes contract arbitrageur (ij)'s optimization, since she would not gain more even if she has chosen a different probability than that of the planner's.

The *ex-ante active* contract arbitrageurs spend all non-money commodities by paying non-money wage z_i and z_j, by receiving $z_i + z_j - q_{ij}(\mathbf{s})$, and by production $q_{ij}(\mathbf{s})$. Therefore, the *ex-ante active* contract arbitrageur hedges all the risk. This is a natural result since contract arbitrageurs are infinitely risk averse in non-money commodities.

The economy with IC_g: By summing up the second dual constraints with arbitrary $Q_{ij}(A, z_i, z_j)$, I get

$$0 \geq \sum_{A \ni (i,j,e_i,e_j)} \left[t_i(A, \Delta z_i) + t_i(A, \Delta z_j) - T_{ij}(A, \Delta z_i, \Delta z_j)\right] Q_{ij}(A, \Delta z_i, \Delta z_j) \qquad [\text{ARB\_}IC_g]$$
$$- \sum_{A \ni (i,j,e_i,e_j)} \sum_{z_i, z_j} \left[\sum_{e_i'} \alpha_i(e_i'|A, z_j) DG_i(e_i'|A, z_i) + \sum_{e_j'} \alpha_j(e_j'|A, z_i) DG_j(e_j'|A, z_j)\right] Q_{ij}(A, z_i, z_j)$$

If $Q_{ij}(A, z_i, z_j) = X_{ij}(A, z_i, z_j)$, the inequality becomes an equality. Moreover, the second line is zero by Lemma 1.

The last two terms are the shadow values of the incentive compatibility constraints. In other words, the seller's profit *internalizes* the shadow value of the incentive compatibility constraints. If $X_{ij}(A, z_i, z_j)$ is the same as the planner's solution, the arbitrageur would get zero profit. If the arbitrageur sells a different amount of the lotteries than the planner's solution, the profit (including the incentive cost represented by the shadow value) would be smaller than zero. Of course, the shadow values of the incentive compatibility constraints are imaginary. A justifying story is that, if the arbitrageur sells a non-incentive compatible lottery, the profit is negative infinity. Even if the arbitrageur sells an incentive compatible contract lottery, the profit would not go up by the following proposition.

Lemma 3 *If $(A, \Delta z_i, \Delta z_j)$ is incentive compatible in the economy with IC_g,*

$$0 \geq \sum_{A \ni (i,j,e_i,e_j)} \sum_{\Delta z_i, \Delta z_j} \left[t_i(A, \Delta z_i) + t_i(A, \Delta z_j) - T_{ij}(A, \Delta z_i, \Delta z_j)\right] Q_{ij}(A, \Delta z_i, \Delta z_j)$$

Therefore, inequality [ARB_IC_g] summarizes contract arbitrageur (ij)'s optimization, since she would not gain more even if she chose a different probability than that of the planner.

The economy with IC_l: By summing up the second dual constraints with arbitrary $Q_{ij}(A, z_i, z_j)$, I get

$$0 \geq \left[t_i(j, e_i, e_j, \Delta z_i) + t_j(i, e_i, e_j, \Delta z_j) - T_{ij}(e_i, e_j, \Delta z_i, \Delta z_j)\right] Q_{ij}(e_i, e_j, \Delta z_i, \Delta z_j) \qquad [\text{ARB\_}IC_l]$$
$$- \sum_{A \ni (i,j,e_i,e_j)} \sum_{z_i, z_j} \left[\sum_{e_i'} \alpha_i(e_i'|j, e_i, e_j, z_j) DG_i(e_i'|A, z_i) + \sum_{e_j'} \alpha_j(e_j'|i, e_i, e_j, z_i) DG_j(e_j'|A, z_j)\right] Q_{ij}(A, z_i, z_j)$$

If $Q_{ij}(A, z_i, z_j) = X_{ij}(A, z_i, z_j)$, the inequality becomes an equality. Moreover, the second line is zero. The interpretation of the last two terms are similar to that of the economy with IC_g.

Lemma 4 *If $(i, j, e_i, e_j, \Delta z_i, \Delta z_j)$ is incentive compatible in the economy with IC_l, then*

$$0 \geq \sum_{\Delta z_i, \Delta z_j} [t_i(j, e_i, e_j, \Delta z_i) + t_j(i, e_i, e_j, \Delta z_j) - T_{ij}(e_i, e_j, \Delta z_i, \Delta z_j)]\, Q_{ij}(e_i, e_j, \Delta z_i, \Delta z_j)$$

Therefore, inequality [ARB_IC_l] summarizes contract arbitrageur (ij)'s optimization, since she would not gain more even if she chose a different probability than that of the planner.

2.3.5 Insurer's Choice: the third dual constraints

The third dual constraint is the insurer's optimization. Again, there is no probability constraint for insurers, i.e. insurers are considered to be perfectly competitive. The third constraint is

$$0 \geq \sum_{(ij) \in A} T_{ij}(A, z_i, z_j) + p(A, \mathbf{s}, (z_k)) \left[\sum_{(ij) \in A} q_{ij}(\mathbf{s}) - \sum_k z_k^{\mathbf{s}} \right] \qquad \text{[INSR]}$$

The insurer behaves differently from ordinary insurers. Insurers are committed to taking care of all the risk involved in the economy. Without commitment, they could have sold more insurance to some firms if the insurance premium exceeds the expected payment to them, and none to the others if not.

The last term is the shadow value of the resource constraint. In other words, the insurer's profit *internalizes* the shadow value of the resource constraint. If $X(A, (z_k))$ is the same as the planner's solution, then the insurer gets zero profit. If the insurer sells insurance that is different from the planner's solution, then the profit (including the incentive cost represented by the shadow value) would be smaller than zero. Of course, the shadow value of the resource constraints are imaginary. A justifying story is that, if the insurer sells insurance that does not satisfy the resource constraint (hence, the insurer cannot deliver what is promised in the insurance contract), the disutility of the insurer is infinite. Even if the insurer sells insurance satisfying the resource constraint, but different from that of the planner's solution, the profit would not go up by the following proposition.

Lemma 5 *If $Q(A, (z_k))$ is the probability that resource constraints are satisfied for all the realizations, then*

$$0 \geq \sum_{(i,j) \in A} T_{ij}(A, z_i, z_j) Q(A, (z_k))$$

Therefore, inequality [INSR] summarizes the insurer's optimization, since she would not gain more even if she chose a different probability than that of the planner.

Example 2 *Suppose there are only two possible teams in the economy with IC_g, (i, j, e_i, e_j) and (k, h, e_k, e_h). In general,*

$$T_{ij}(A, \Delta z_i, \Delta z_j) X_{ij}(A, \Delta z_i, \Delta z_j) \neq \sum_z \sum_{\mathbf{s}} p(A, \mathbf{s}, z)[z_i^{\mathbf{s}} + z_j^{\mathbf{s}} - q_{ij}(\mathbf{s})] X_{ij}(A, z_i, z_j)$$
$$T_{kh}(A, \Delta z_k, \Delta z_h) X_{kh}(A, \Delta z_k, \Delta z_h) \neq \sum_z \sum_{\mathbf{s}} p(A, \mathbf{s}, z)[z_k^{\mathbf{s}} + z_h^{\mathbf{s}} - q_{kh}(\mathbf{s})] X_{kh}(A, z_k, z_h)$$

In other words, the insurance premium $T_i(A, \Delta z_i, \Delta z_j)$ does not correctly reflect the expected value of the payment from the insurer to the insured. Say $T_i(A, \Delta z_i, \Delta z_j) > \sum_z \sum_{\mathbf{s}} p(A, \mathbf{s}, z)[z_i^{\mathbf{s}} + z_j^{\mathbf{s}} - q_{ij}(\mathbf{s})]X_{ij}(A, z_i, z_j)$. If the insurer has access to a contingent claims market where she can purchase contingent claims $z_i^{\mathbf{s}} + z_j^{\mathbf{s}} - q_{ij}(\mathbf{s})$ at price $\sum_z \sum_{\mathbf{s}} p(A, \mathbf{s}, z)[z_i^{\mathbf{s}} + z_j^{\mathbf{s}} - q_{ij}(\mathbf{s})]X_{ij}(A, z_i, z_j)$, she would not sell insurance to (kh), but only to (ij). In other words, unless the risk premium correctly reflects the expected value of future payments, commitment of the insurer is critical in proving the welfare theorems.

One might wonder if it would be possible to pick a weight profile λ so that the insurance premium reflects the expected value of the payment from insurance. However, the number of possible matching structures is in general larger than that of individuals, so it would not be feasible to adjust λ in order to make all the insurance premiums reflect the true values of the insurance.

Discussion on the equality of insurance premium and the expected value of future payment is revisited when I compare finite and continuum economies in terms of the convergence of the former to the latter.

2.3.6 Public randomization device: How to Exercise Random Contracts

Lastly, a public randomization device is required in order to fully decentralize the planner's solution. For a realized uncertainty $\mathbf{s}_A$, firms pay randomized payoff $z_i^{\mathbf{s}}$ with probability $X_i(A, z_i)/\sum_{\tilde{z}_i} X_i(A, \tilde{z}_i)$. Moreover, all the firms' exercise of random contracts are correlated by the primal constraints.

$$X_{ij}(A, z_i, z_j) - \sum_{z_{-i}, z_{-j}} X(A, (z_k)) = 0 \ , \quad X_i(A, z_i) - \sum_{z_j} X_{ij}(A, z_i, z_j) = 0$$

In other words, even after the realization of state $\mathbf{s}_A$, firms exercise random contracts by observing the public randomization device. If the randomization device realizes $(z_k)_{k \in I \cup J}$ with probability $X(A, (z_k))/\sum_{(\tilde{z}_k)} X(A, (\tilde{z}_k))$ when $(A, \mathbf{s}_A)$ is realized, firms pay $z_k^{\mathbf{s}}$, i.e. coordinated exercise of contracts.

If it is assumed that a reputable institution announces the outcome of the public randomization device, the story of public randomization can be justified. However, the information that the institution has to process is enormous. So the story might not conform to the true spirit of welfare theorems or Arrow-Debreu, in which small economic entities observe a very small set of information (prices) compared to the size of the economy (allocations). Moreover, in order for the insurer to bid for the license, she needs to know many details of the economy.

There are two ways to deal with the existence of a public randomization device: one is the lack of commitment and the other is competition. In the model, firms serve as a commitment technology in the sense that individuals joining a firm are committed to the consumption paid by the firms; hence, they are committed not to participate in the spot market after the realization of all uncertainty ($A, \mathbf{s}_A$, and the realization of the randomization device). If I assume away the assumption of the commitment technology of firms, spot prices would serve as a public randomization device. The illustration is given in section 6.4. The other way to go around the existence of the randomization device is competition. The model here has a finite number of individuals. In section 3, I propose a continuum model and show convergence of the finite model to the continuum model, in which the economy-wide public randomization device can be 'decentralized' into firm-specific randomization devices; hence, the public randomization device is eliminated.

2.4 Comment on Combined Welfare Theorems

In the classical private good exchange economy, only one point on the contract curve can be decentralized without money transfers among individuals. However, Theorem 1 shows that all the efficient allocations of my model can be decentralized. The difference comes from the fact that commodity price $p(A,(z_k),\mathbf{s})$ is non-linear in the sense that it is a function of (z_k). In the classical exchange economy, price is not a function of allocation even though price is determined to clear the commodity market.

The non-linearity of the price is illustrated by the linear programming formulation. Makowski and Ostroy (1996) consider the classical exchange economy and write down the resource constraint as

$$\sum_i \sum_z z x_i(z) = 0$$

$x_i(z)$ is the individual measure that is in the objective function in the planner's problem and in the resource constraint at the same time. Therefore, $x_i(z)$ directly connects the utility function and the resource constraint; hence, the formulation enabled them to reflect the concavity of the utility function in price, i.e. linear price. On the other hand, in the finite model of this paper, analogues of the resource constraint of the classical exchange economy are the probability consistency constraints and the resource constraint, which are

$$X_i(A,z_i) - \sum_{z_j} X_{ij}(A,z_i,z_j) = 0, \quad X_{ij}(A,z_i,z_j) - \sum_{z_{-i},z_{-j}} X(A,(z_k)) = 0$$

$$\left[\sum_k z_k^{\mathbf{s}_A} - \sum_{(i,j)\in A} q_{ij}(\mathbf{s})\right] X(A,(z_k)) = 0, \forall \mathbf{s}_A \in \mathbf{S}_A, A, (z_k)$$

It is impossible to write down the resource constraint using measures on individuals' or active contracts' behavior. Therefore, an economy-wide measure $X(A,(z_k))$ is used. Since the dual value of the resource constraints represent the price of the commodities in the linear programming formulation, the commodity price becomes a function of allocation – non-linear price. Also, the existence of the economy-wide measure means that the corresponding dual constraint has to be interpreted as an economic agent's maximization problem. The fact that $X(A,(z_k))$ describes the details of allocations in the economy is the main reason that the economic agent 'insurer' has to take care of all the risks in the economy.

Through measure $X(A,(z_k))$, unlike Makowski and Ostroy (1996), here individual constraints become redundant. Formally,

$$\sum_A \sum_{z_i} X_i(A, z_i) = 1 \text{ implies } \sum_A \sum_{z_j} X_j(A, z_j) = 1$$

since

$$\sum_A \sum_{z_i} X_i(A, z_i) = \sum_A \sum_{z_i} \sum_{z_{-i}} X(A, (z_k)) = \sum_A \sum_{z_j} \sum_{z_{-j}} X(A, (z_k)) = \sum_A \sum_{z_j} X_j(A, z_j).$$

It is well known that redundant constraints in linear program make it possible to give arbitrary dual values to the constraints with the sum of the dual values being constant. Since individual probabilities are redundant in the linear programming formulation, the planner can give arbitrary dual values of the individual probability constraints, which are the payoffs to the individuals. Therefore, the planner's problem with arbitrary weight can be decentralized.

In summary, the existence of the economy-wide measure, non-linearity of commodity prices, and the redundancy of individuals' probability constraints are equivalent. The economy-wide measure forces the decentralized economy to have an insurer for efficiency, and the redundancy of the individuals' probability constraints makes it possible to decentralize the entire utility frontier. Considering that the insurer's role is essentially to cross-subsidize because the insurer has to take care of all the risks in the economy, the combined welfare theorem is not surprising.

Ostroy and Song (2005) characterize best correlated equilibria (in the sense that the weighted sum of utilities are maximized) as Lindahl equilibria where sellers of correlated equilibria constrained by the incentive compatibility condition are introduced to the game. The finite model in this paper can be interpreted as an extended version of correlated equilibria (see Myerson (1991)) of games in which each team does not observe the suggested team structure of others. The uncertainty of the entire matching structure relaxes the incentive compatibility constraints more. The result of the combined welfare theorems is also presented in Ostroy and Song (2005).

3 A Matching Problem with Moral Hazard: Continuum Economy

A version of the contractual matching model with continuum individuals is proposed. In section 4, the relationship between the models in section 2 and 3 is discussed.

Having continuums of individuals means that uncertainty is 'certain' in the sense that gross uncertainty does not exist due to the *law of large numbers*. So, I assume that the contract is only a function of idiosyncratic shock. However, the assumption is not without question. In the finite model, the contract is a function of economy-wide shocks. So, the logical extension of the contractual form in the continuum model is that the contract is a function of the distribution of realized shocks. This discrepancy between the intuitive contractual form (a function of idiosyncratic shock) and the contractual form logically extended from the finite model is discussed and reconciled.

Also, the linear programming formulation of the resource constraint by the firms' probabilities is possible because of the assumption of the contractual form and the *law of large numbers*. In the finite model, the formulation of the resource constraint is possible only thorugh $X(A,(z_k))$, but not by $X_{ij}(A,z_i,z_j)$. However, since gross uncertainty is absent in the continuum model, the firms' probabilities could be used for the formulation of the resource constraint. The consequence of this formulation is a tighter link between consumption and the value of firm. Insurance premium fails to reflect the expected value of payments from the insurance in the finite model, but it is recovered in this continuum model.

3.1 Planner's Problem

The planner's problem is formulated as a linear program.

Assignments of Matching and Efforts: There are two kinds of continuum individual with finite types, I and J. A typical type in I, J, or $I \cup J$ is denoted by i, j, or k. Each individual can work only when each is matched with someone from the other population. When i and j are matched, it is said that *firm* (ij) *is formed*. When they are matched, they contract upon their efforts, which are denoted by $(e_i, e_j) \in \mathcal{E}^2$. When individual i is not matched with anybody, it is denoted that individual i is matched to $\emptyset$.

Matching: I write (i, j, e_i, e_j) if i and j are matched with efforts (e_i, e_j). If i is not matched with anybody, I write $(i, \emptyset)$.

Technology, state, and matching: The state of firm (i, j, e_i, e_j) is denoted by $s \in S = \{1, \ldots, S\}$. s is realized with probability $\varphi(s; e_i, e_j)$. In other words, the realization of state s is effected by efforts (e_i, e_j).

The output for individual firm (i, j, e_i, e_j) at state s is $q(s)$.

Commodities, Allocation, and Utility function: There are L non-money commodities and one money commodity. Let Δz_i^s be a randomized assignment of consumption at state s. Let $\Delta z_i := (\Delta z_i^s)_{s \in S}$ be a randomized assignment of consumption for all states. If there is no randomness in consumption, z_i is defined by $z_i := (z_i^s)_{s \in S}$.

Assignment/Allocation is formally defined.

Definition 4 *Assignment/Allocation* $(i, j, e_i, e_j, \Delta z_i, \Delta z_j)$ *specifies* [**Assignment**] *which individuals* (i, j) *are matched with which efforts* (e_i, e_j)*, and* [**Allocation**] *what the randomized consumption at the realization of state* s *of firm* (i, j, e_i, e_j) *are.*

Let $x_i(z_i | j, e_i, e_j, \Delta z_i)$ be the marginal probability of z_i for random contract $(i, j, e_i, e_j, \Delta z_i, \Delta z_j)$. The expected utility of i is denoted by $\sum_{s \in S} \sum_{z_i^s} v_i(z_i^s) \varphi(s; e_i, e_j) x_i(z_i | e_i, e_j) - E_i(e_i)$ where $E_i(e_i)$ represents the utility cost of effort e_i. $v_i(\cdot)$ is strictly increasing, strictly concave, and differentiable.

Notation of a few probabilities and measures: Let us define a few measures and/or probabilities to better understand the economy.

$x_{ij}(e_i, e_j, z_i, z_j)$**:** fraction of firm (i, j, e_i, e_j) in the economy where i and j consume z_i and z_j

$x_i(j, e_i, e_j, z_i)$**:** fraction/probability of type i that is matched to j with (e_i, e_j), and consumes z_i

$x_i(\emptyset, z_i)$**:** fraction/probability of type i that is not matched with anybody, and consumes z_i

$x_i(z_i|j, e_i, e_j, \Delta z_i)$**:** fraction/probability of type i consuming z_i conditional on matching (i, j, e_i, e_j) with random consumption Δz_i, $x_i(j, e_i, e_j, z_i)/\sum_{\tilde{z}_i} x_i(j, e_i, e_j, \tilde{z}_i)$

Note that $x_i(j, e_i, e_j, z_i)$ and $x_i(z_i|j, e_i, e_j, \Delta z_i)$ can be interpreted as probabilities because of the assumption of a continuum of individuals of a type.

For individual i, random matching with efforts and consumption is described by probability $x_i(j, e_i, e_j, z_i)$. For example, if $x_i(j, e_i, e_j, z_i) = x_i(j, e_i, e_j, z_i') = x_i(j, e_i', e_j', z_i'') = x_i(j', e_i'', e_j'', z_i''') = x_i(\emptyset, z_i'''') = 1/5$, individual i is matched to j with (e_i, e_j) and probability 2/5, to j with (e_i', e_j') and probability 1/5, to j' with (e_i'', e_j'') and probability 1/5, or with nobody with probability 1/5. In the case that (i, j, e_i, e_j) is realized, i consumes z_i or z_i' each with probability 1/2. In the case that individual i is matched to j (or j') with (e_i', e_j') (or (e_i'', e_j'')), individual i consumes z_i'' (or z_i''') with certainty. If i is not matched with anybody, i consumes z_i''''.

$x_i(j, e_i, e_j, z_i)$ is a marginal fraction of $x_{ij}(e_i, e_j, z_i, z_j)$ by $x_i(j, e_i, e_j, z_i) = \sum_{z_j} x_{ij}(e_i, e_j, z_i, z_j)$. As in section 2, the measures/fractions are considered as if there is only a finite support.

Choice of Information Structure: The choice of information structure does not matter in the continuum model, since wage is a function of only idiosyncratic shock.

3.1.1 Linear Programming Formulation of Planner's Problem

Objective Function: The following is the expected utility for individual i.

$$U_i(X_i(\cdot, \cdot)) := \sum_{j, e_i, e_j} \sum_{z_i} \left[\sum_{s \in S} v_i(z_i^s) \varphi(s; e_i, e_j) - E_i(e_i) \right] x_i(j, e_i, e_j, z_i) + \sum_{z_i} v_i(z_i) x_i(\emptyset, z_i)$$

since $x_i(j, z_k, e_i, e_j)$ is the probability that the planner puts individual i in matching (i, j, e_i, e_j) with consumption z_i.

$$\sum_i \lambda_i U_i(x_i(\cdot)) + \sum_j \lambda_j U_j(x_j(\cdot))$$

Probability constraints: Sum of probabilities $x_i(\cdot)$ equal to 1.

$$\sum_{j, e_i, e_j} \sum_{z_i} x_i(j, e_i, e_j, z_i) + \sum_{z_i} x_i(\emptyset, z_i) = 1, \forall i \in I \cup J$$

Matching constraints: $x_i(j, e_i, e_j, z_i)$ cannot be arbitrary. Let $x_{ij}(e_i, e_j, z_i, z_j)$ be the fraction that i and j are matched with efforts (e_i, e_j) and consumption z_i and z_j. Then the following must hold.

$$x_i(j, e_i, e_j, z_i) = \sum_{z_j} x_{ij}(e_i, e_j, z_i, z_j), \forall i, j, e_i, e_j, z_i$$

Resource Constraint: Since all the uncertainty is washed out by the *law of large numbers*, the resource constraint can be written as

$$\sum_{i,j}\sum_{e_i,e_j}\sum_{z_i,z_j}\sum_{s}[z_i^s + z_j^s - q(s)]\varphi(s;e_i,e_j)x_{ij}(e_i,e_j,z_i,z_j) + \sum_{k\in I\cup J}\sum_{z_k} z_k x_k(\emptyset, z_k) \le 0$$

Incentive Compatibility Constraints:

$$DG_i(e_i'|j,e_i,e_j,z_i) := \left[\sum_s v_i(z_i^s)\varphi(s;e_i',e_j) - E_i(e_i')\right] - \left[\sum_s v_i(z_i^s)\varphi(s;e_i,e_j) - E_i(e_i)\right],$$

then the incentive compatibility condition is

$$\sum_{z_i} DG_i(e_i'|e_i,e_j,z_i)x_{ij}(e_i,e_j,z_i,z_j) \le 0$$

In summary, the planner's problem with weight profile $(\lambda_k)_{k\in I\cup J}$ is

$$\begin{aligned}
(P)\quad \max\quad & \sum_i \lambda_i U_i(x_i(\cdot)) + \sum_j \lambda_j U_j(x_j(\cdot)) \\
s.t.\quad & \sum_{j,e_i,e_j}\sum_{z_i} x_i(j,e_i,e_j,z_i) + \sum_{z_i} x_i(\emptyset, z_i) = 1, \forall i, \text{ similarly for } j \\
& x_i(j,e_i,e_j,z_i) - \sum_{z_j} x_{ij}(e_i,e_j,z_i,z_j) = 0, \forall i,j,e_i,e_j,z_i \\
& \sum_{i,j}\sum_{e_i,e_j}\sum_{z_i,z_j}\sum_{s}[z_i^s + z_j^s - q(s)]\varphi(s;e_i,e_j)x_{ij}(e_i,e_j,z_i,z_j) + \sum_{k\in I\cup J}\sum_{z_k} z_k x_k(\emptyset, z_k) \le 0 \\
& \sum_{z_i} DG_i(e_i'|e_i,e_j,z_i)x_{ij}(e_i,e_j,z_i,z_j) \le 0 \\
& x_i(j,e_i,e_j,z_i), x_i(\emptyset,z_i), x_{ij}(e_i,e_j,z_i,z_j) \ge 0
\end{aligned}$$

Assumption 4 *The domain of the planner's problem is not empty.*

Proposition 5 *A solution of the planner's problem exists.*

Proof. From the assumption on $v_i(\cdot)$, the non-emptiness of the domain, and Carathéodory's Theorem on convex hull, a maximum exists. ∎

Note that the solution of the planner's problem is incentive-constrained efficient by definition.

Definition 5 *A random assignment of firms and consumption is* incentive constrained *efficient if the assignment solves the planner's problem.*

3.2 Decentralization of Efficient Assignment

The dual linear programming of the planner's program specifies a market environment where the decentralized economy replicates the optimal solution of the planner. The answer includes commodities, prices of the commodities, timing of relevant markets for the commodities, necessary commitment, technologies such as randomization devices, and arbitrageurs of the commodities. Detailed derivation of the market environment for the decentralization is provided in section 3.3, and I state market environment here without derivation.

Commodities and Prices: Lottery commodities on contracts specify who are matched, what the random payoffs are, and what the efforts are. Prices of the lotteries are individual-specific, i.e. Lindahl prices denoted by $\tau_i(j, e_i, e_j, \Delta z_i)$ for $(i, j, e_i, e_j, \Delta z_i)$.

Players of the economy: Besides the individuals in the planner's problem, perfectly competitive contract arbitrageurs are derived from the dual linear program.

Contract arbitrageurs are expected money maximizers. They specialize in contract writing. They maximize expected money income by selling lotteries on matching (i, j, e_i, e_j) to individuals, and by innovating a contract. Although they are risk neutral in terms of money, they get a payoff of negative infinity if they do not fulfill the contract. Since their supplies are unlimited, they end up with zero profit in equilibrium.

Markets and Timing of Markets: There are a market for lotteries on contracts and an ex-post market of commodities. The timing is described below.

The 0th stage (*ex-ante* phase): Individuals and contract arbitrageurs trade lotteries on contract $(i, j, e_i, e_j, \Delta z_i)$. Contract arbitrageurs trade in a contingent claims market to hedge the risk from non-money commodities.

The 1st stage (*interim* phase): Matchings are realized by a public randomization device.

The 2nd stage: i and j choose efforts.

The 3rd stage: Idiosyncratic shocks are realized, the contracts are exercised, the unmatched and contract arbitrageurs trade, and individuals consume. The ex-post market is cleared by the delivery of contract arbitrageurs' contingent claims and the spot trade of the unmatched.

Commitment: In the 3rd stage, agents who are matched do not trade.

Existence of randomization devices: In order to exercise random contracts, randomization devices are required. However, unlike in section 2, there is no need to have a centralized public randomization device for the exercise of contract. Instead, it is sufficient for each firm (i, j, e_i, e_j) to possess a randomization device that they use for its own contract.

Definition 6 (Definition of Equilibrium) **1. Individual Optimization:** *Individuals trade lotteries on contract $(i, j, e_i, e_j, \Delta z_i)$ at per-unit probability price of $\tau_i(j, e_i, e_j, \Delta z_i)$. Once contract $(i, j, e_i, e_j, \Delta z_i, \Delta z_j)$ is realized, individuals i and j choose efforts. After the realization of idiosyncratic shocks and the randomization devices, contracts are exercised, and individuals consume. Formally, individuals' problems* *are*

$$\max_{e_i'} \max_{\xi_i(\cdot)} \sum_{j,e_i,e_j} \left\{ \sum_{z_i} \left[\sum_{\mathbf{s}} v_i(z_i^s)\varphi(s;e_i',e_j) - E_i(e_i') \right] x_i(z_i|j,e_i,e_j,\Delta z_i) \right\} \xi_i(j,e_i,e_j,\Delta z_i) + \sum_{z_i} v_i(z_i)\xi_i(\emptyset,z_i)$$

$$s.t. \sum_{(j,e_i,e_j,\Delta z_i)} \tau_i(j,e_i,e_j,\Delta z_i)\xi_i(j,e_i,e_j,\Delta z_i) + p\sum_{z_i} z_i\xi_i(\emptyset,z_i) = 0$$

2. **Contract-arbitrageur's Optimization:** *Unlimitedly supplied contract-arbitrageurs trade lotteries at the price of* $\tau_i(j,e_i,e_j,\Delta z_i)$. *Once the contract arbitrageur's contract is picked by the randomization device, she takes charge of the firm. Formally, the contract arbitrageurs' problems are*

$$\max_{\xi_{ij}(e_i,e_j,\Delta z_i,\Delta z_j)} \sum_{e_i,e_j} \sum_{\Delta z_i,\Delta z_j} \left[\tau_i(j,e_i,e_j,\Delta z_i) + \tau_i(j,e_i,e_j,\Delta z_j) - \mathcal{T}_{ij}(e_i,e_j,\Delta z_i,\Delta z_j)\right] \xi_{ij}(e_i,e_j,\Delta z_i,\Delta z_j)$$

$$s.t.\ (i,j,e_i,e_j,\Delta z_i) \text{ is incentive compatible.}$$

$$\text{where } \mathcal{T}_{ij}(e_i,e_j,\Delta z_i,\Delta z_j) = p\sum_{z_i,z_j} \left\{ \sum_s [z_i^s + z_j^s - q(s)]\varphi(s;e_i,e_j) \right\} \xi_{ij}(z_i,z_j|e_i,e_j)$$

3. **Clearance of Commodity Market:** *The commodity market clears.*

$$\sum_{i,j}\sum_{e_i,e_j}\sum_{z_i,z_j}\sum_s [z_i^s + z_j^s - q(s)]\varphi(s;e_i,e_j)x_{ij}(e_i,e_j,z_i,z_j) + \sum_{k\in I\cup J} z_k x_k(\emptyset,z_k) \le 0$$

4. **Matching Market Clearance (Consistency):** *Matching market clears in the sense that lottery purchases are consistent across the population.*

$$x_i(j,e_i,e_j,\Delta z_i) = \sum_{\Delta z_j} x_{ij}(e_i,e_j,\Delta z_i,\Delta z_j)$$

Theorem 2 (Welfare Theorems) [**The first welfare theorem**] *A price-taking equilibrium with a lottery trade and randomization devices is constrained efficient.* [**The second welfare theorem**] *There exists a weight profile* λ *such that the planner's assignment of matching and allocation of consumption (constrained by the incentive compatibility constraint) can be decentralized by a lottery trade and the public randomization devices.* [**Characterization**] *Contract arbitrageurs get zero profit.*

Discussion follows after the proof.

3.3 Characterization of Equilibria and Proof of Theorem 2

The dual linear programming of the planner's linear program is derived. From the dual constraints (the constraints of the dual linear program), a proper definition of price-taking equilibrium is derived.

3.3.1 Dual Linear Programming

Let dual variables corresponding to each constraint of the planner's linear program be y_i, $\tau_i(j, e_i, e_j, z_i)$, $\tau_j(i, e_i, e_j, z_j)$, p, and $\alpha_i(e_i'|j, e_i, e_j, z_j)$. Then the following is the dual linear program.

$$
\begin{aligned}
(D) \quad \min \quad & \sum_i y_i + \sum_j y_j \\
s.t. \quad & y_i \geq \lambda_i \left[\sum_s v_i(z_i^s)\varphi(s; e_i, e_j) - E_i(e_i) \right] - \tau_i(j, e_i, e_j, z_i) \\
& y_i \geq \lambda_i v_i(z_i) - p z_i \\
& 0 \geq \tau_i(j, e_i, e_j, z_i) + \tau_j(i, e_i, e_j, z_j) + p \sum_s [q(s) - z_i^s - z_j^s]\varphi(s; e_i, e_j) \\
& \qquad - \sum_{e_i'} \alpha_i(e_i'|j, e_i, e_j, z_j) DG_i(e_i'|e_i, e_j, z_i) - \sum_{e_j'} \alpha_j(e_j'|i, e_i, e_j, z_i) D_j(e_j'|e_i, e_j, z_j) \\
& p, \alpha_i(e_i'|j, e_i, e_j, z_i) \geq 0
\end{aligned}
$$

Proposition 6 (Fundamental Theorem of Linear Programming) *(1) There exists a solution for the primal linear program. (2) There exists a solution for the dual linear program. (3) The values of the primal and dual programs are the same.*

Proposition 7 (Complementary Slackness)

$$
\begin{aligned}
& x_i(j, e_i, e_j, z_i) \left\{ y_i - \left[\lambda_i \left(\sum_s v_i(z_i^s)\varphi(s; e_i, e_j) - E_i(e_i) \right) - \tau_i(j, e_i, e_j, z_i) \right] \right\} = 0 \\
& x_i(\emptyset, z_i) \left\{ y_i - \lambda_i v_i(z_i) - p z_i \right\} = 0 \\
& x_{ij}(e_i, e_j, z_i, z_j) \left\{ \tau_i(j, e_i, e_j, z_i) + \tau_j(i, e_i, e_j, z_j) + p \sum_s [q(s) - z_i^s - z_j^s]\varphi(s; e_i, e_j) \right. \\
& \qquad \left. - \sum_{e_i'} \alpha_i(e_i'|j, e_i, e_j, z_j) DG_i(e_i'|e_i, e_j, z_i) - \sum_{e_j'} \alpha_j(e_j'|i, e_i, e_j, z_i) D_j(e_j'|e_i, e_j, z_j) \right\} = 0
\end{aligned}
$$

Before analyzing and interpreting the meaning of dual constraints, I prove two lemmas that can be shown by a direct application of the fundamental theorem of LP.

Lemma 6 *At the optimal solution of linear program,*

$$
\begin{aligned}
y_i = & \lambda_i \sum_{j, e_i, e_j} \sum_{z_i} \left[\sum_s v_i(z_i^s)\varphi(s; e_i, e_j) - E_i(e_i) \right] x_i(j, e_i, e_j, z_i) + \lambda_i \sum_{z_i} v_i(z_i) x_i(\emptyset, z_i) \\
& - \sum_{j, e_i, e_j} \sum_{z_i} \tau_i(j, e_i, e_j, z_i) x_i(j, e_i, e_j, z_i) - p \sum_{z_i} z_i x_i(\emptyset, z_i) \\
0 = & \sum_{z_i} \tau_i(j, e_i, e_j, z_i) x_i(j, e_i, e_j, z_i) + \sum_{z_j} \tau_j(i, e_i, e_j, z_j) x_j(i, e_i, e_j, z_j) \\
& + p \sum_{z_i, z_j} \sum_s [q(s) - z_i^s - z_j^s]\varphi(s; e_i, e_j) x_{ij}(e_i, e_j, z_i, z_j)
\end{aligned}
$$

Lemma 7 *There exist a weight profile $\lambda := (\lambda_k) \geq 0$ such that*

$$\sum_{j,e_i,e_j} \sum_{z_i} \tau_i(j,e_i,e_j,z_i)x_i(j,e_i,e_j,z_i) + p\sum_{z_i} z_i x_i(\emptyset, z_i) = 0, \forall i \in I \cup J.$$

I assume the following.

Assumption 5 *There exists $\lambda \gg 0$ satisfying Lemma 7.*

Mas-Collel (1985) found a sufficient condition for the existence of $\lambda \gg 0$ in the classical exchange economy. A similar condition is derived and it replaces Assumption 5 in appendices.

3.3.2 Characterization of Decentralization

The planner's problem can be interpreted as a revenue maximization problem as in section 2.

3.3.3 Individual Choice: the first dual constraint

The dual variable of the first constraint of the primal linear program, y_i, is the value of individual i to the planner. Define prices using the optimal values of the dual linear program.

$$\tau_i(j,e_i,e_j,\Delta z_i) := \sum_{z_i} \tau_i(j,e_i,e_j,z_i)\xi_i(z_i|j,e_i,e_j,\Delta z_i)$$

$\tau_i(j,e_i,e_j,\Delta z_i)$ is the price of a random contract that (i) realizes matching (i,j,e_i,e_j) with probability $\sum_{z_i} \frac{\xi_i(j,e_i,e_j,z_i)}{\sum_{\tilde{j},z_i,\tilde{e}_i,\tilde{e}_j} \xi_i(\tilde{j},\tilde{e}_i,\tilde{e}_j,z_i)}$ and (ii) gives consumption z_i^s at state s with probability $\sum_{\tilde{z}_i^{\mathbf{s}}:\tilde{z}_i=z_i^{\mathbf{s}}} \frac{\xi_i(j,e_i,e_j,\tilde{z}_i)}{\sum_{\tilde{z}_i} \xi_i(j,e_i,e_j,\tilde{z}_i)}$.

By summing up the first dual constraints with arbitrary $\xi_i(j,e_i,e_j,z_i)$,

$$\begin{aligned} y_i \geq & \lambda_i \sum_{j,e_i,e_j} \sum_{z_i} \left[\sum_{s\in S} v_i(z_i^s)\varphi(s;e_i,e_j) - E_i(e_i) \right] \xi_i(j,e_i,e_j,z_i) + \lambda_i \sum_{z_i} v_i(z_i)\xi_i(\emptyset,z_i) \qquad \text{[indv]} \\ & - \sum_{j,e_i,e_j} \sum_{z_i} \tau_i(j,e_i,e_j,z_i)\xi_i(j,e_i,e_j,z_i) - p\sum_{z_i} z_i \xi_i(\emptyset,z_i) \end{aligned}$$

$$\begin{aligned} \Leftrightarrow\ y_i/\lambda_i \geq & \sum_{j,e_i,e_j,\Delta z_i} \left\{ \sum_{z_i} \left[\sum_{s\in S} v_i(z_i^s)\varphi(s;e_i,e_j) - E_i(e_i) \right] \xi_i(z_i|j,e_i,e_j,\Delta z_i) \right\} \xi_i(j,e_i,e_j,\Delta z_i) + \lambda_i \sum_{z_i} v_i(z_i)\xi_i(\emptyset,z_i) \\ & - \frac{1}{\lambda_i} \left[\sum_{(j,e_i,e_j,\Delta z_i)} \tau_i(j,e_i,e_j,\Delta z_i)\xi_i(j,e_i,e_j,\Delta z_i) + p\sum_{z_i} z_i \xi_i(\emptyset,z_i) \right] \end{aligned}$$

Again, if $\xi_i(j,e_i,e_j,z_i) = x_i(j,e_i,e_j,z_i)$, the inequalities become equalities.

From Lemma 7, pick λ such that $\sum_{j,e_i,e_j} \sum_{z_i} \tau_i(j,e_i,e_j,z_i)x_i(j,e_i,e_j,z_i) + p\sum_{z_i} z_i x_i(\emptyset,z_i) = 0$. Then,

$$\sum_{j,e_i,e_j,\Delta z_i} t_i(j,e_i,e_j,\Delta z_i)x_i(j,e_i,e_j,\Delta z_i) + p\sum_{z_i} z_i x_i(\emptyset,z_i) = 0.$$

The meaning of the equality in Lemma 2 is that money expenditure on the lottery purchase is zero if the purchase is the same as that of the planner's solution. Therefore, the above inequality [indv] summarizes individual i's optimization, since, if individual i has chosen a different probability than that of the planner's, the purchase of the different probability would be infeasible or suboptimal. Also, y_i/λ_i is interpreted as the *ex-ante* utility of i before realization of (i, j, e_i, e_j). Moreover, $1/\lambda_i$ is i's marginal utility of income. In other words, if individual i were given ϵ amount of money in the beginning, he could have increased his expected utility by changing his purchase of lotteries.

The utility after contract $(i, j, e_i, e_j, \Delta z_i)$ is realized is

$$\frac{y_i}{\lambda_i} + \frac{1}{\lambda_i} t_i(j, e_i, e_j, \Delta z_i) x_i(j, e_i, e_j, \Delta z_i) = \sum_{z_i} \left[\sum_{s \in S} v_i(z_i^s) \varphi(s; e_i, e_j) - E_i(e_i) \right]$$

In general the second term of the left-hand side of the equality is not zero. Constrained efficiency typically requires agents of the same type to obtain different utility levels when assigned to different teams.

3.3.4 Contract Arbitrageurs' Choice: the second dual constraints

The second dual constraint is the contract arbitrageurs' optimization. There is no probability constraint for arbitrageurs, i.e. arbitrageurs are a freely available input to the planner. Being freely available, arbitrageurs are considered to be perfectly competitive.

The contract arbitrageur pays non-money commodity z_i and z_j to i and j, which are not necessarily same as $q_{ij}(s)$. To fill the gap between $z_i + z_j$ and $q_{ij}(s)$, the contract arbitrageur will trade in the contingent claims market.

By summing up the second dual constraints with arbitrary $\xi_{ij}(e_i, e_j, z_i, z_j)$,

$$\begin{aligned}
0 \geq & \sum_{z_i} \tau_i(j, e_i, e_j, z_i) \xi_i(j, e_i, e_j, z_i) + \sum_{z_j} \tau_j(i, e_i, e_j, z_j) \xi_j(i, e_i, e_j, z_j) \\
& - p \sum_{z_i, z_j} \sum_{s} [z_i^s + z_j^s - q_{ij}(s)] \varphi(s; e_i, e_j) \xi_{ij}(e_i, e_j, z_i, z_j) \qquad \text{[arb]} \\
& - \sum_{z_i, z_j} \left[\sum_{e_i'} \alpha_i(e_i'|e_i, e_j, z_j) DG_i(e_i'|e_i, e_j, z_i) + \sum_{e_j'} \alpha_j(e_j'|e_i, e_j, z_i) DG_j(e_j'|e_i, e_j, z_j) \right] \xi_{ij}(e_i, e_j, z_i, z_j) \\
\Rightarrow \quad 0 \geq & \tau_i(e_i, e_j, \Delta z_i) \xi_i(j, e_i, e_j, \Delta z_i) + \tau_i(e_i, e_j, \Delta z_j) \xi_j(i, e_i, e_j, \Delta z_j) \\
& - p \sum_{z_i, z_j} \left\{ \sum_{s} [z_i^s + z_j^s - q_{ij}(s)] \varphi(s; e_i, e_j) \xi_{ij}(z_i, z_j | e_i, e_j, \Delta z_i, \Delta z_j) \right\} \xi_{ij}(e_i, e_j, \Delta z_i, \Delta z_j) \\
& - \sum_{z_i, z_j} \left[\sum_{e_i'} \alpha_i(e_i'|e_i, e_j, z_j) DG_i(e_i'|e_i, e_j, z_i) + \sum_{e_j'} \alpha_j(e_j'|e_i, e_j, z_i) DG_j(e_j'|e_i, e_j, z_j) \right] \xi_{ij}(e_i, e_j, z_i, z_j)
\end{aligned}$$

If $\xi_{ij}(e_i, e_j, z_i, z_j) = x_{ij}(e_i, e_j, z_i, z_j)$, the inequality becomes an equality. Moreover, the last line is zero by Lemma 6.

The last line reflects the shadow value of the incentive compatibility constraints. The interpretation is same to that of section 2.3.4, so it is omitted here.

Lemma 8 *If* $(i, j, e_i, e_j, \Delta z_i, \Delta z_j)$ *is incentive compatible, then*

$$\begin{aligned} 0 \quad \geq \quad & \tau_i(e_i, e_j, \Delta z_i)\xi_i(j, e_i, e_j, \Delta z_i) + \tau_i(e_i, e_j, \Delta z_j)\xi_j(i, e_i, e_j, \Delta z_j) \\ & -p \sum_{z_i, z_j} \left\{ \sum_s [z_i^s + z_j^s - q_{ij}(s)]\varphi(s; e_i, e_j)\xi_{ij}(z_i, z_j|e_i, e_j, \Delta z_i, \Delta z_j) \right\} \xi_{ij}(e_i, e_j, \Delta z_i, \Delta z_j) \end{aligned}$$

Therefore, inequality [arb] summarizes contract arbitrageur (ij)'s optimization, since she would not gain more even if she chose a different probability than that of the planner. To summarize, the contract arbitrageur purchase non-money commodity $z_i^{\mathbf{s}} + z_j^{\mathbf{s}} - q_{ij}(s)$ in the contingent claims market $\varphi(s; e_i, e_j)p$.

3.3.5 The randomization devices: How to Exercise Random Contracts

Unlike the finite model, contracts are not correlated since the wage function depend only on idiosyncratic shock, which makes it possible not to have an economy-wide randomization device. Instead, each firm possesses its own randomization device for the exercise of random contracts, and all of the devices are independent.

3.4 Comment on the Welfare Theorems

The combined welfare theorem in the finite model was possible because of the redundancy of the individuals' probability constraints. The probability constraint of one individual implied all other probability constraints; hence, decentralization was possible for the entire utility frontier. In the continuum model, there is no such extreme redundancy. However, a redundancy still exists in some cases. Considering that a firm consists of two individual (i.e. a firm is a public good), a possible redundancy is not surprising.

The following is an extreme case similar to the combined welfare theorem in the finite model.

Example 3 $I = \{i\}$ *and* $J = \{j\}$*. Suppose the planner's solution is such that individuals are always matched with certain contracts. Individual i's probability constraint implies j's probability constraint since*

$$\sum_{e_i, e_j} \sum_{z_i} x_i(j, z_i, e_i, e_j) = \sum_{e_i, e_j} \sum_{z_j} \sum_{z_i} x_{ij}(z_i, z_j, e_i, e_j) = \sum_{e_i, e_j} \sum_{z_j} x_j(i, z_j, e_i, e_j)$$

Therefore, the entire utility frontier can be decentralized. Formally, for any weight (λ_i, λ_j)*, suppose*

$$\sum_{e_i, e_j} \sum_{z_i} t_i(j, z_i, e_i, e_j)x_i(j, z_i, e_i, e_j) \neq 0 \quad \textit{and} \quad \sum_{e_i, e_j} \sum_{z_i} t_i(j, z_i, e_i, e_j)x_i(j, z_i, e_i, e_j) \neq 0$$

Define

$$\begin{aligned} \hat{y}_i \quad &:= \quad y_i + \sum_{\tilde{e}_i, \tilde{e}_j} \sum_{\tilde{z}_i} t_i(j, \tilde{z}_i, \tilde{e}_i, \tilde{e}_j)x_i(j, \tilde{z}_i, \tilde{e}_i, \tilde{e}_j) \\ \hat{t}_i(j, z_i, e_i, e_j) \quad &:= \quad t_i(j, z_i, e_i, e_j) - \sum_{\tilde{e}_i, \tilde{e}_j} \sum_{\tilde{z}_i} t_i(j, \tilde{z}_i, \tilde{e}_i, \tilde{e}_j)x_i(j, \tilde{z}_i, \tilde{e}_i, \tilde{e}_j) \end{aligned}$$

and similarly for $\hat{y}_j$ and $\hat{t}_j(\cdot)$. Then, all the dual constraints are satisfied with the newly defined dual values, and they are also optimal for the dual linear program. Therefore, it is shown that the entire utility frontier can be decentralized.

In a larger economy, the redundancy is not as extreme as in Example 3 even when the redundancy exists.

Example 4 *$I = \{1, \ldots, N\}$ and $J = \{1, \ldots, N\}$. The individual probability constraints of $I \setminus \{i\}$ and J imply the probability constraint of i. Formally,*

$$\sum_{j,e_i,e_j} \sum_{z_i} x_i(j, z_i, e_i, e_j) = \sum_{j,e_i,e_j} \sum_{z_i,z_j} x_{ij}(z_i, z_j, e_i, e_j) = \sum_{j} \sum_{e_i,e_j} \sum_{z_j} x_j(i, z_j, e_i, e_j) \quad \text{for } i$$

$$1 \geq \sum_{j,e_{\tilde{i}},e_j} \sum_{z_{\tilde{i}}} x_{\tilde{i}}(j, z_{\tilde{i}}, e_{\tilde{i}}, e_j) = \sum_{j,e_{\tilde{i}},e_j} \sum_{z_{\tilde{i}},z_j} x_{\tilde{i}j}(z_{\tilde{i}}, z_j, e_{\tilde{i}}, e_j) = \sum_{j} \sum_{e_{\tilde{i}},e_j} \sum_{z_j} x_j(\tilde{i}, z_j, e_{\tilde{i}}, e_j) \quad \text{for } \tilde{i} \in I \setminus \{i\}$$

Adding up the last terms, I get

$$\sum_{j} \sum_{\tilde{i} \in I, e_{\tilde{i}}, e_j} \sum_{z_j} x_j(\tilde{i}, z_j, e_{\tilde{i}}, e_j) \leq N$$

Therefore, the following has to be true for i.

$$\sum_{j,e_i,e_j} \sum_{z_i} x_i(j, z_i, e_i, e_j) \leq 1$$

Again redundancy of the constraints of the primal linear program implies multiple dual solutions. Multiple dual solutions mean that $T(\lambda)$ defined in the proof of Lemma 7 is not a singleton in general. Therefore, there are multiple λ satisfying Lemma 7 since $T(\cdot)$ is upper-semi continuous.

Example 3 shares a picture similar to a double auction model with one buyer and one seller. Suppose there are one type of buyer with valuation of unity and one type of seller with cost of production being zero. Equilibrium price p could be anything in $[0, 1]$. The utility frontier is represented by $\{(y_b, y_s)|y_b + y_s = 1\}$, and any point can be decentralized by a certain price $p \in [0, 1]$.

Example 4 shares a picture similar to a double auction model with many buyers and many sellers. Suppose the set of buyer types is $\{1, 0.6, 0\}$, and the set of seller types is $\{0, 0.4, 1\}$. The utility frontier is represented by $\{(y_{b1}, y_{b.6}, y_{b0}, y_{s0}, y_{s.4}, y_{s1})| \sum_b y_b + \sum_s y_s = 1.2\}$. It is not possible to decentralize the entire utility frontier. Only $\{(y_{b1}, y_{b.6}, y_{b0}, y_{s0}, y_{s.4}, y_{s1})|y_{b1} = 1 - p, y_{b.6} = 0.6 - p, y_{b1} = 0, y_{s0} = p, y_{s.4} = p - 0.4, y_{s1} = 0, p \in [0.4, 0.6]\}$ can be decentralized.

However, cases similar to the above two examples generically do not happen. Each example has one mass of each type, and all individuals are matched. If there is a positive fraction of a type that is unmatched, then the redundancy of individual probability constraints cannot be derived. Alternatively, suppose that an economy is picked with a mass of each type randomly drawn from independent uniform distribution [0.5, 1.5]. Generically, not all individuals would be matched. In that case, the redundancy of individual probability cannot be derived either.

3.5 Efficiency and Lotteries

The realized utility after the realization of contract $(i, j, e_i, e_j, \Delta z_i)$ is

$$\frac{y_i}{\lambda_i} + \frac{1}{\lambda_i} t_i(j, e_i, e_j, \Delta z_i) x_i(j, e_i, e_j, \Delta z_i) = \sum_{z_i} \left[\sum_{s \in S} v_i(z_i^s) \varphi(s; e_i, e_j) - E_i(e_i) \right]$$

In general the second term of the left-hand side of the equality is not zero. Therefore, if non-degenrate lotteries are used, individual k's payoffs are different across realizations of $(i, j, e_i, e_j \Delta z_i)$. However, each individual's expected utility is y_k/λ_k. Therefore, efficiency typically requires agents of the same type to obtain different utilities when assigned to different matching.

Proposition 8 *1. In general, trades of lotteries are necessary for constrained efficiency.*

2. Therefore, efficiency typically requires individuals of the same type to obtain different expected utility when assigned to different firms.

3. Compensating wage differentials which equalize the utilities of individuals in different firms are generally incompatible with efficiency.

Similar results in a different setting are reported in Bennardo and Piccolo (2005) that study a general equilibrium model where agents' preferences, productivity and labor endowments depend on their health status. Among their results are (i) efficiency typically requires agents of the same type to obtain different expected utilities if assigned to different occupations, (ii) competitive equilibria are first-best if lottery contracts are enforceable, buy typically not if only assets with deterministic payoffs are traded, and (iii) compensating wage differentials which equalize the utilities of workers in different jobs are incompatible with ex-ante efficiency. The above result in this paper is another confirmation of their result in a team formation environment.

The essential reason that the lotteries are used is due to the indivisibility nature of a team. The lotteries are beneficial in continuum models when same type individuals get different utilities depending on the realization of their different occupations. Without the difference of utilities, lotteries are degenerate. Ellickson, Grodal, Scotchmer, Zame (1999, 2001) (henceforth, EGSZ) consider an economy where a lottery market is not allowed. Therefore, even though when the same type individuals work in different occupation, their utilities have to be the same for the existence of equilibrium. On the other hand, Cole and Prescott (1997) consider an economy where lotteries can be traded. Through the lottery and expected utility functions, the economy can improve efficiency. The following simple example shows the basic difference of the two models.

Example 5 *There are two ex-ante identical individuals. The only way they can produce is to form a team, in which one becomes principal and the other becomes agent. The output of non-money commodities from* *the team is 4. The utility functions are given by*

$$Agent: \ln(x+1) - \ln 2 \quad , \quad Principal: \ln(x+1)$$

The allocation that gives the same utilities to ex-ante identical principal and agent is that the agent gets 3 and the principal gets 1. Since they are ex-ante identical the welfare can be measured by the sum of their utilities, which is $\ln 4$*.*

On the other hand, they can agree on a lottery by which one becomes the principal with consumption 2, and the other becomes the agent with consumption 2. The welfare of this economy is $2(\ln 3 - \frac{1}{2}\ln 2)$*, which is larger than* $\ln 4$*.*

The first allocation of the planner in the example is congruent to that of EGSZ (1999, 2001), while the second lottery allocation of the planner is congruent to that of Cole and Prescott (1997) and this paper.

One might think that the economy of EGSZ does not utilize the gains from playing lotteries, and the economy of Cole and Prescott does. However, I argue that they just have different data of economies. The data of EGSZ is preference over bundles, and the data of Cole and Prescott is preference over risks.

There are many ways to represent a preference over bundles with a utility function. For example, the following utility function represents the same preference over occupations and consumption to that of the above example.

$$\text{Agent}: 1 - e^{-\frac{x-1}{2}} \quad , \quad \text{Principal}: 1 - e^{-x}$$

Therefore, the equilibrium not using lotteries is the same. However, the equilibrium using lotteries is different. In the equilibrium, lottery with the new utility function is that the ex-ante identical individual becomes *Principal* who consumes 1.4621 with half probability, or becomes *Agent* who consumes 2.5379 with half probability, since $\frac{d}{dx}\left(1 - e^{-\frac{x-1}{2}}\right)|_{x=2.5379} = \frac{d}{dx}\left(1 - e^{-x}\right)|_{x=1.4621}$.

It is well known that expected utility (preference over risk) is unique up to affine transformation.[2] In other words, once a utility function (not preference) is given, preference over risk is given. EGSZ (1999, 2001) consider economies in which the data is preference over bundles, while Cole and Prescott (1997) consider an economy in which the data is preference over risky assets. Makowski and Ostroy (2003) and Rahman (2005a) follow the second criterion of efficiency in terms of utilities from non-money commodities, but quasi-linearity of utilities makes the market for lotteries on matching vanish since money is used to compensate the difference of utilities from non-money commodities when same type individuals are assigned to different occupations.

[2]Non-unique representation of preference over risk due to affine transformation is not important. Suppose that $v_i(z)$ is replaced with $av_i(z) + b$ in the economy. The constant term b does not change anything except that the value of the planner's objective function shifts up. Since decentralization is obtained by adjusting λ, following the method of Shapley (1969), the effect of a is neutralized in the process of finding the right λ.

3.6 Non-linearity and Linearity of Prices for Lotteries: Arbitrage Opportunity

As is in Cole and Prescott (1997), the lottery commodity space can be linearized by the infinite dimensional vector space of non-negative measures on a compact set. Formally, individual i's feasible set is the set of non-negative measure defined on $J \times \mathcal{E}^2 \times \mathbb{R}^{|S|}$, $\mathbf{M}[J \times \mathcal{E}^2 \times \mathbb{R}^{|S|}]$. The proper price of a lottery is a continuous function(al) defined on $\mathbf{M}[J \times \mathcal{E}^2 \times \mathbb{R}^{|S|}]$ with weak-$*$ topology.

The linearity of price in the infinite dimensional space is necessary when decentralization is considered. Without linearity of the lottery price, there could be an arbitrage opportunity in the lottery trading. For example, an arbitrageur can create a new lottery by going long and short for some lotteries. In order for the arbitrage to be unsuccessful in the equilibrium, the price of lottery has to be linearly decomposed in the infinite linear space.

Even though the price of the lottery is linear in the space of probabilities, the price is not linear in the space of commodities in the sense that the price of the lottery cannot be linearly decomposed into commodity prices. By the same logic in the previous paragraph, there would exist an arbitrage opportunity without the exclusive nature of contracts. In other words, exclusiveness of contracts is another requirement for decentralization derived from the dual linear program.

4 Convergence of the Finite Model to the Continuum One

In the finite model, contracts depend on the economy-wide shock. To the best of my knowledge, continuum models without a global shock have assumed that contracts depend only upon an idiosyncratic shock. In the continuum model, the uncertainty outside of a firm is not 'uncertain' anymore since all uncertainty wash out because of the continuum assumption. However, it is not clear how the discrepancy of the contractual forms in the finite and continuum models could be connected.

4.1 Elimination of the Public Randomization Device

In the finite economy in section 2, the contract is a function of the shocks of all teams present in the economy. When I consider convergence of the finite model to the continuum, the natural extension of the contractual form for the continuum model is a function of the distribution of realized shocks. For example, suppose that there is only one ex-ante identical team of $[0, 1]$ in the continuum economy. Suppose each team can have only two states 'bad:=0' and 'good:=1' with probability 1/3 and 2/3. The wage function is $w : F \to \mathbb{R}^L$ where $F = \{f : [0, 1] \to \{0, 1\} | m(f^{-1}[\{0\}]) = 1/3, m(f^{-1}[\{1\}) = 2/3\}$ where $m(\cdot)$ is Lebesque measure.[3] In other words, the general form of wage $w(\cdot)$ is a function of the distribution of realized shocks. More simply, let the

[3] f is not a measurable function in general with Lesbeque measure defined on $[0, 1]$. Therefore, F is not even defined. But, Uhlig (1996) shows how to deal with this kind of non-measuarability.

name of a team be $t \in [0, 1)$. A possible contractual form is $w(s_t, s_{t+0.1})$ where s_t is the idiosyncratic shock of team t and $s_{t+0.1}$ is the shock of team $t + 0.1$ (modulus 1). The potential gain of this contractual form is to relax the incentive compatibility condition by using the randomness outside of firm t. (See Mookherjee (1984)). However, section 3 focuses on a special form of contracts that are assumed to be a function of only the idiosyncratic shock.

I, however, prove that the restriction on the contractual form is without loss of generality. Formally, let V^C be the value of the continuum model from the solution of the planner. Let $V^F(1)$ be the value of the finite model. Let $V^F(n)$ be the value of the n-times replicated finite model divided by n. If $V^C = \lim_{n\to\infty} V^F(n)$ can be shown, the restriction on the contractual form is without loss of generality when the interest is on efficiency.

Since contractual forms in the finite economy are more general than those in the continuum, the relaxation of incentive compatibility constraints in the limit of the finite economy will be no worse than that in the continuum. Therefore

$$\lim_{n\to\infty} V^F(n) \geq V^C$$

I show that the inequality turns out to be an equality.

Theorem 3 *The continuum model with contracts depending only on the idiosyncratic shock is a limit of the finite model with contracts depending on the economy-wide shock.*

Proof. Assume the converse, $\lim_{n\to\infty} V^F(n) > V^C$, which means that there is a large number N such that $V^F(N) > V^C$. In terms of $V^F(N)$, for a given A, let $\mathbf{s}$ and $\tilde{\mathbf{s}}$ be such that

$$q_{ij}(\mathbf{s}) = q_{ij}(\tilde{\mathbf{s}}), \quad \sum_{(i,j)\in A} q_{ij}(\mathbf{s}) = \sum_{(i,j)\in A} q_{ij}(\tilde{\mathbf{s}}) \text{ but } a = z_i^{\mathbf{s}} \neq z_i^{\tilde{\mathbf{s}}} = b \text{ in } X_i(A, z_i)$$

Suppose that there are no such $\mathbf{s}$ and $\tilde{\mathbf{s}}$ for other $k \neq i$, and there are only two such $\mathbf{s}$ and $\tilde{\mathbf{s}}$ for i, which is a special case. The proof for the general case is in the appendices.

Define $\tilde{X}(A, (z'_k))$, and $\tilde{X}(A, (z''_k))$ such that

$$\begin{aligned}
&z_i'^{\mathbf{t}} = z_i''^{\mathbf{t}} = z_i^{\mathbf{t}} \text{ if } \mathbf{t} \neq \mathbf{s}, \tilde{\mathbf{s}} \\
&z_i'^{\mathbf{t}} = a, z_i''^{\mathbf{t}} = b \text{ if } \mathbf{t} = \mathbf{s}, \tilde{\mathbf{s}} \\
&z'_k = z''_k = z_k \text{ if } k \neq i \\
&\tilde{X}(A, (z'_k)) = \frac{\Pr_A^{\mathbf{s}}}{\Pr_A^{\mathbf{s}} + \Pr_A^{\tilde{\mathbf{s}}}}, \ \tilde{X}(A, (z'_k)) = \frac{\Pr_A^{\tilde{\mathbf{s}}}}{\Pr_A^{\mathbf{s}} + \Pr_A^{\tilde{\mathbf{s}}}} X(A, (z_k))
\end{aligned}$$

Then the following is derived.

$$\tilde{X}(A, (z'_k)) + \tilde{X}(A, (z''_k)) = X(A, (z_k))$$

Also the definitions of $\tilde{X}_i(A, z'_i)$, $\tilde{X}_i(A, z''_i)$, $\tilde{X}_{ij}(A, z'_i, z_j)$, and $\tilde{X}_{ij}(A, z'_i, z_j)$ are followed so that the probability clearing constraints hold for the new variables.

$$\tilde{X}_i(A, z'_i) + \tilde{X}_i(A, z''_i) = X_i(A, z_i), \ \tilde{X}_{ij}(A, z'_i, z_j) + \tilde{X}_{ij}(A, z''_i, z_j) = X_{ij}(A, z_i, z_j)$$

Now, the probabilities are defined to satisfy probability clearing conditions. Also, the incentive compatibility constraints and the value of the primal are the same by the following equalities.

$$v_i(a)[\Pr^{\mathbf{s}}_A + \Pr^{\tilde{\mathbf{s}}}_A]\tilde{X}_i(A, z_i') + v_i(b)[\Pr^{\mathbf{s}}_A + \Pr^{\tilde{\mathbf{s}}}_A]\tilde{X}_i(A, z_i'') = [v_i(a)\Pr^{\mathbf{s}}_A + v_i(b)\Pr^{\mathbf{s}'}_A]X_i(A, z_i)$$

$$v_i(a)[\Pr^{\mathbf{s}}_A + \Pr^{\tilde{\mathbf{s}}}_A]\tilde{X}_{ij}(A, z_i', z_j) + v_i(b)[\Pr^{\mathbf{s}}_A + \Pr^{\tilde{\mathbf{s}}}_A]\tilde{X}_{ij}(A, z_i'', z_j) = [v_i(a)\Pr^{\mathbf{s}}_A + v_i(b)\Pr^{\tilde{\mathbf{s}}}_A]X_{ij}(A, z_i, z_j)$$

$$v_i(a)[\Pr^{\mathbf{s}}_{A'} + \Pr^{\tilde{\mathbf{s}}}_{A'}]\tilde{X}_{ij}(A, z_i', z_j) + v_i(b)[\Pr^{\mathbf{s}}_{A'} + \Pr^{\tilde{\mathbf{s}}}_{A'}]\tilde{X}_{ij}(A, z_i'', z_j) = [v_i(a)\Pr^{\mathbf{s}}_{A'} + v_i(b)\Pr^{\tilde{\mathbf{s}}}_{A'}]X_{ij}(A, z_i, z_j)$$

where A' is the same as A except that $(i, j, e_i', e_j) \in A'$ instead of $(i, j, e_i', e_j) \in A$.

Although the resource constraint is not satisfied in general, the following expected resource constraint holds.

$$\left[\sum_k z_k'^{\mathbf{s}} - \sum_{(i,j)\in A} q_{ij}(\mathbf{s})\right]\tilde{X}(A, (z_k'))\Pr^{\mathbf{s}}_A + \left[\sum_k z_k'^{\tilde{\mathbf{s}}} - \sum_{(i,j)\in A} q_{ij}(\tilde{\mathbf{s}})\right]\tilde{X}(A, (z_k'))\Pr^{\tilde{\mathbf{s}}}_A$$
$$+ \left[\sum_k z_k''^{\mathbf{s}} - \sum_{(i,j)\in A} q_{ij}(\mathbf{s})\right]\tilde{X}(A, (z_k''))\Pr^{\mathbf{s}}_A + \left[\sum_k z_k''^{\tilde{\mathbf{s}}} - \sum_{(i,j)\in A} q_{ij}(\tilde{\mathbf{s}})\right]\tilde{X}(A, (z_k''))\Pr^{\tilde{\mathbf{s}}}_A = 0$$

Now interpret the model as the continuum version of $V^F(N)$. Decompose the continuum economy into a continuum of identical infinitesimal sub-economies. For each infinitesimal sub-economy, assign the randomized team structure and consumption by the new assignment $\tilde{X}(\cdot)$ described above. For each sub-economy, the resource constraint does not hold, but by transferring resources across sub-economies, the resource constraint holds as a whole economy. Apparently, the assignment $\tilde{X}(\cdot)$ could be suboptimal for the continuum economy. Therefore,

$$V^C = V^C(N) \geq V^F(N)$$

where the value of the planner's problem in this continuum economy is $V^C(N)$. Now a contradiction is derived since $V^F(N) > V^C$ was assumed. ∎

Therefore, it is without loss of generality to consider the contractual form depending only on idiosyncratic shock in the continuum model. Uhlig (1996) raises a similar question. He asks what would be a limit model of a finite model where the allocation is dependent on the economy-wide shock. This paper is an answer to the question in a specific economy.

Even though the proof is done by contradiction, it has the flavor of a constructive proof. In the proof, if there are different assignments of consumption at state $\mathbf{s}$ and $\tilde{\mathbf{s}}$ for individual i in team (i, j), by increasing the randomness of contracts in each state $\mathbf{s}$ and $\tilde{\mathbf{s}}$, the dependency of assignments on the economy-wide shock is reduced to that of an idiosyncratic shock s_{ij}. In doing so, the randomness of contracts inside firms becomes more involved. In other words, the randomness of contracts inside firms in the continuum model is not only for the relaxation of incentive compatibility, but also for the simplicity of contracts. Even though the contract becomes complicated because the random contract depends on the firm's randomization device, observability of the economy-wide shock becomes unnecessary. Considering that the spirit of *the price-taking equilibrium* is that an economic entity (an individual in the exchange economy, or a team in this paper) can decide one's action by observing only local information such as price (instead of the entire economy), this simplification is more faithful to the spirit of decentralization.

Since the dependency of contracts on others' shock is eliminated, the public randomization device correlating exercises of contracts is not necessary anymore in the continuum model. Therefore, a corollary of Theorem 3 is followed.

Corollary 1 *In the continuum model as a limit of the finite model, the (centralized) public randomization device for the exercise of contracts is unnecessary. Each firm has its own randomization device, and their devices are independent.*

4.2 Comparison of Welfare Theorems: Decentralizable Utility Frontier

Theorem 1 states that the entire utility frontier can be decentralized in the finite economy, while Theorem 2 states that is not the case in the continuum economy. The difference seems to contradict Theorem 3 that the continuum model is a limit of the finite model.

In the proof of Lemma 2, prices are shifted up or down uniformly so that (1) individuals' expenditures become zero, (2) individuals' choices are not effected, and (3) contract arbitrageurs' and the insurer's choices are not effected. However, in doing so, no attention was given to the relationship between the insurance premium and the expected payments from the insurance. For example, the following equality was not considered in the economy with IC_g.

$$T_{ij}(A, \Delta z_i, \Delta z_j) = \sum_{(z_k)} \sum_{\mathbf{s} \in \mathbf{S}_A} p(A, \mathbf{s}, (z_k))[z_i^{\mathbf{s}} + z_j^{\mathbf{s}} - q_{ij}(\mathbf{s})] \Pr(\mathbf{s}; A) \frac{X(A, (z_k))}{\sum_{(\tilde{z}_k)} X(A, (\tilde{z}_k))}$$

where the left-hand side is the insurance premium and the right-hand side is the expected value of the payment from the insurance.

It is generically impossible to pick λ to satisfy the equalities for all the insurance purchased in equilibrium, since there are too many equalities to be satisfied and the number of control variables is only $|I| + |J| - 1$, the effective number of (λ_i). For example, suppose $I = \{1, 2\}$, $J = \{1, 2\}$, and there are four ways of assignments A_1, A_2, A_3, A_4. Note that matching specifies efforts too. Then, the number of the equalities that has to be satisfied is 4, while the number of the control variables is 3. Therefore, the equalities would be impossible in general.

4.3 Cream-skimming and Cross-subsidy of the Insurer

The roles of an insurer in the finite economy are (1) to pool risk in the economy as much as incentive compatibility constraints permit and (2) to cross-subidize across teams. Those two roles require the commitment of the insurer not to 'cream-skim'. Cream-skimming in this paper differs from the same word in the literature on insurance market under asymmetric information. Cream-skimming in Rothschild and Stiglitz (1976) describes a behavior of an insurer trying to seek less risky individuals under the adverse selection problem.

There is no adverse selection problem in the model. I define cream-skimming behavior to be when an insurer tries to sell more insurance to some teams because the premium of that particular insurance is larger than the expected payments of that insurance.

In the finite economy, there is typically a contract arbitrageur (i, j, e_i, e_j) who is active in the equilibrium with contract $(\Delta z_i, \Delta z_j)$ such that

$$T_{ij}(A, \Delta z_i, \Delta z_j) > \sum_{(z_k)} \sum_{\mathbf{s}\in\mathbf{S}_A} p(A, \mathbf{s}, (z_k))[z_i^{\mathbf{s}} + z_j^{\mathbf{s}} - q_{ij}(\mathbf{s})] \Pr(\mathbf{s}; A) \frac{X(A, (z_k))}{\sum_{(\tilde{z}_k)} X(A, (\tilde{z}_k))}.$$

Therefore, as soon as the insurer breaks her promise to take care of all the risk involved in the economy, she can gain positive profit by selling insurance only to firm (i, j, e_i, e_j). So, the commitment to no cream-skimming and/or exclusion of a cream-skimming insurer are necessary. However, if the equivalence of the insurance premium and the expected payments were possible, then there is no room for the insurer to cream-skim. Since the difference between the premium and the expected payments of insurance is the amount of the cross-subsidy, the conditions of no cross-subsidy and no profit from cream-skimming are equivalent.

In the continuum economy, the role of the insurer is redundant because the ex-post market can do the work of the insurer. Market price and expected payments of $(i, j, e_i, e_j, \Delta z_i, \Delta z_j)$ are identical to

$$p \sum_{z_i, z_j} \left\{ \sum_s [z_i^s + z_j^s - q_{ij}(s)] \varphi(s; e_i, e_j) \xi_{ij}(z_i, z_j | e_i, e_j, \Delta z_i, \Delta z_j) \right\}.$$

In other words, there is no cross-subsidy; hence, there is no profit from cream-skimming.

Proposition 9 *1. Cross-subsidies across firms vanish in the continuum economy.*

2. Cream-skimming is not possible in the continuum model.

Similar phenomena are reported in other literatures. For example, Ellickson, Grodal, Scotchmer, and Zame (2001) find that equilibrium may not exist in their model since club memberships are indivisible and choices of club memberships must be coordinated across the population. However, for large finite club economies, an approximate core can be approximately decentralized. Another example in a much more different context is Ju (2005) which studies risk sharing problems in a finite number of states. He shows that there is no strategy-proof mechanism under aggregate uncertainty, while there is a mechanism without aggregate uncertainty. My model shares a similar phenomenon in the sense that, since there is aggregate uncertainty on the matching structure in the finite economy, decentralization is impossible without the insurer avoiding cream-skimming. However, in the continuum model where there is no aggregate uncertainty on the matching structure, decentralization is possible without the insurer.

4.4 Linearity and Non-linearity of Commodity Price: Arrow-Debreu and Lindahl Prices

In the finite economy, commodity price $p(A, \mathbf{s}, (z_k))$ is non-linear in the sense that the price function has an argument of allocation (z_k). In the continuum economy, the commodity price p is linear. The difference comes from how the resource constraint is written: economy-wide measure $X(A, \mathbf{s}, (z_k))$ is used to describe the resource constraint in the finite economy, while individual measures $x_i(\cdot)$ are used to describe the resource constraint. Because the individual probabilities are used to describe the resource constraint in the continuum economy, tighter link between all the teams' optimization and resource constraint is possible in the continuum economy, while each team's optimization is connected only through insurer to the resource constraint in the finite economy. This characteristic of the finite economy makes the transaction of each team not anonymous (an Lindahl price of the insurance premium) since each team could access the commodity market only through an insurer. However, each team accesses the commodity market directly in the continuum economy, so that the commodity prices could be Arrow-Debreu ones.

5 Team model

Only teams composed of two members have been considered so far. The framework studied in the previous sections can be easily extended to study team models. I present only the setups. All results are derived in similar ways, so they are omitted.

5.1 Finite model

Population and Efforts: A typical individual is denoted by $i \in N$. *Team*, T, is an element in $2^N \setminus \{\emptyset\}$. The efforts vector is denoted by $e_T = (e_i)_{i \in T} \in \mathcal{E}^T$.

Team Contractual Matching Function: The team contractual matching function is

$$\begin{aligned} A \quad &: \quad N \to \{(T, e_T) | T \in 2^N \setminus \{\emptyset\}, e_T \in \mathcal{E}^{|T|}\} \cup \{\emptyset\} \\ s.t. \quad & A(i) = (T, e_T) \to A(j) = (T, e_T), \forall j \in T \end{aligned}$$

If i is not matched to anybody, $A(i) = \emptyset$. For the simplicity of notation, it is often written $(T, e_T) \in A$ if $A(i) = (T, e_T), \forall i \in T$. I sometimes write $T \in A$ when T is formed in matching A with a certain e_T. Define set $\mathcal{A}$ to be the set of all the team contractual matching functions, and let $\mathcal{A}_{(T, e_T)} = \{A | A(i) = (T, e_T), \forall i \in T\}$. The team structure realized in the economy is a partition of N.

Technology, state, and matching: The state of team T is denoted by $s_T \in S = \{1, \ldots, S\}$. For a given matching function A, the state of the economy is $\mathbf{s}_A = (s_T)_{T \in A}$. $\mathbf{s}_A$ is realized with probability $\Pr(\mathbf{s}_A; A)$.

The output for individual firm at state s_T is $q(s_T)$.

Commodities, Allocation, and Utility Function: There are L non-money commodities and one money commodity.

Definition 7 *Assignment/Allocation* $(T, (z_i)_{i\in T}, e_T)$ *specifies* **[Assignment]** *who are involved* ($i \in T$) *with which efforts* (e_T) *and* **[Allocations]** *what are the (possibly randomized) payoffs at the realization of the state of economy,* $(A, \mathbf{s})$ *where* $A \in \mathcal{A}_{(T,e_T)}$ *and* $\mathbf{s} \in \mathbf{S}_A$

The expected utility of i for given matching A is denoted by $\sum_{\mathbf{s}\in\mathbf{S}_A} v_i(z_i^{\mathbf{s}})\Pr(\mathbf{s}; A) - E_i(e_i)$ where $E_i(e_i)$ represents the monetary cost of effort e_i. $v_i(\cdot)$ is strictly increasing, strictly concave, and differentiable.

Timing: The timing is similarly defined as in section 2.

Notations of a few probabilities: Identically defined as in section 2.

Choice of Information Structure: Two information structures are identically defined as in section 2.

Assumption 6 (*Local* Information Structure) *Once T is formed with a contract implementing* (e_T), *individuals do NOT observe how others are matched.*

Assumption 7 (*Global* Information Structure) *Once T is formed with contract* $(z_i)_{i\in T}$ *implementing* e_T, *individuals DO observe how others are matched.*

Allowing random contracts, the *local incentive compatibility constraint* (IC_l) is

$$\sum_{A\ni(T,e_T)} \sum_{z_T} DG_i(e_i'|A, z_i)X_T(A, z_T) \leq 0 \qquad [IC_l]$$

where deviation gain $DG_i(e_i'|A, z_i)$ is defined as

$$DG_i(e_i'|A, z_i) := \left[\sum_{\mathbf{s}\in\mathbf{S}_A} v_i(z_i^{\mathbf{s}})\Pr(\mathbf{s}; A|e_i') - E_i(e_i')\right] - \left[\sum_{\mathbf{s}\in\mathbf{S}_A} v_i(z_i^{\mathbf{s}})\Pr(\mathbf{s}; A) - E_i(e_i)\right]$$

The *global incentive compatibility constraint* (IC_g) is

$$\sum_{z_T} DG_i(e_i'|A, z_i)X_T(A, z_T) \leq 0 \qquad [IC_g]$$

No private access to contingent market: It is assumed the same way as in section 2.

5.2 Primal and Dual Linear Programming for Finite Economy

The primal linear program is the following.

$$
\begin{aligned}
(P) \quad \max \quad & \sum_i \sum_A \sum_{z_i} \lambda_i \left[\sum_{\mathbf{s} \in \mathbf{S}_A} v_i(z_i^{\mathbf{s}}) \Pr(\mathbf{s}; A) - E_i(e_i) \right] X_i(A, z_i) \\
s.t. \quad & \sum_A \sum_{z_i} X_i(A, z_i) = 1, \forall i \in N \\
& X_i(A, z_i) - \sum_{z_{-i}} X_T(A, z_T) = 0, \forall z_i, A \\
& X_T(A, z_T) - \sum_{z_{-T}} X(A, (z_i)) = 0, \forall T, z_T, A \\
& \left[\sum_i z_i^{\mathbf{s}} - \sum_{T \in A} q_T(\mathbf{s}) \right] X(A, (z_i)) \le 0, \forall \mathbf{s} \in \mathbf{S}_A, A \\
& \sum_{z_i} DG_i(e_i'|A, e_T, z_i) X_T(A, z_T) \le 0, \forall i \in T, z_T, A \\
& X_i(A, z_i), X_T(A, z_T), X(A, (z_k)) \ge 0
\end{aligned}
$$

The dual linear program is the following.

$$
\begin{aligned}
(D) \quad \min \quad & \sum_i y_i \\
s.t. \quad & y_i \ge \lambda_i \left[\sum_{\mathbf{s} \in \mathbf{S}_A} v_i(z_i^{\mathbf{s}}) \Pr(\mathbf{s}; A) - E_i(e_i) \right] - t_i(A, z_i) \\
& 0 \ge \sum_{i \in T} t_i(A, z_i) - t_T(A, z_T) - \sum_{i \in T} \sum_{e_i'} \alpha_i(e_i'|A, (z_j)_{j \in T \setminus \{i\}}) DG_i(e_i'|A, z_i) \\
& 0 \ge \sum_{T \in A} t_T(A, z_T) + p(A, \mathbf{s}, (z_k)) \left[\sum_{T \in A} q_T(\mathbf{s}) - \sum_k z_k \right] \\
& \alpha_i(\cdot), p(\cdot) \ge 0
\end{aligned}
$$

5.3 Continuum Model

The continuum model of team matching is almost identical to that of section 3.

Population and Efforts: A typical individual is denoted by $i \in N$. *Team*, T, is an element in $2^N \setminus \{\emptyset\}$. The efforts vector is $e_T = (e_i)_{i \in T} \in \mathcal{E}^T$.

Assignment of Matching and Efforts: Matching is denoted by (T, e_T), where $T \in N$ and $e_T = (e_i)_{i \in T}$

Technology, state, and matching: The state of team T is $s_T \in S = \{1, \ldots, S\}$ realized with probability $\varphi(s; e_T)$. The output of firm T is $q_T(s)$.

Notation of a few probabilities

$x_T(e_T, z_T)$**:** the fraction of firm (T, e_T) in the economy where $i \in T$ consumes z_i

$x_i(T, e_T, z_i)$**:** the fraction/probability of type i that belongs to team T with e_T, and consumes z_i

$x_i(z_i|T, e_T, \Delta z_i)$**:** the fraction of type i consuming z_i conditional on matching (T, e_T) with random consumption Δz_i

Commodities, Assignment/Allocation, and Utility Function: There are L non-money commodities and one money commodity. Assignment/allocation is the following.

Definition 8 *Assignment/allocation $(T, (z_i)_{i\in T}, e_T)$ specifies* [**Assignment**] *who are involved ($i \in T$) with which efforts (e_T) and* [**Allocation**] *what are the randomized payoffs at the realization of state of economy, s.*

The expected utility of i for given matching (T, e_T) is denoted by $\sum_{s\in S} v_i(z_i^s)\varphi(s; e_T) - E_i(e_i)$.

5.4 Primal and Dual Linear Programming for Continuum Economy

The primal linear program is the following.

$$
\begin{aligned}
(P) \quad \max \quad & \sum_i \sum_{T\ni i} \sum_{z_i} \lambda_i \left[\sum_s v_i(z_i^s)\varphi(s; e_T) - E_i(e_i) \right] x_i(T, e_T, z_i) + \sum_i \sum_{z_i} v_i(z_i) x_i(\emptyset, z_i) \\
s.t. \quad & \sum_{T\ni i} \sum_{e_T} \sum_{z_i} x_i(T, e_T, z_i) + \sum_{z_i} x_i(\emptyset, z_i) = 1, \forall i \in N \\
& x_i(T, e_T, z_i) - \sum_{z_{-i}} x_T(e_T, z_T) = 0, \forall i, T, z_i, e_T \\
& \sum_T \sum_{e_T} \sum_{z_T} \sum_s [\sum_{i\in T} z_i^s - q_T^s]\varphi(s; e_T) x_T(e_T, z_T) + \sum_i \sum_{z_i} z_i x_i(\emptyset, z_i) \leq 0 \\
& \sum_{z_i} DG_i(e_i'|T, e_T, z_i) x_T(e_T, z_T) \leq 0, \forall i \in T, z_T, e_T \\
& x_i(T, e_T, z_i), x_T(e_T, z_T) \geq 0
\end{aligned}
$$

The dual linear program is the following.

$$
\begin{aligned}
(D) \quad \min \quad & \sum_i y_i \\
s.t. \quad & y_i \geq \lambda_i \left[\sum_s v_i(z_i^s)\varphi(s; e_T) - E_i(e_i) \right] - t_i(T, e_T, z_i), \forall i, T, z_i, e_T \\
& y_i \geq \lambda_i v_i(z_i) - p z_i \\
& 0 \geq \sum_{i\in T} t_i(T, e_T, z_i) - p \sum_s [\sum_{i\in T} z_i^s - q(s)]\varphi(s; e_T) - \sum_{i\in T} \sum_{e_i'} \alpha_i(e_i'|T, e_T, z_{T\setminus\{i\}}) DG_i(e_i'|T, e_T, z_i) \\
& \alpha_i(\cdot), p \geq 0
\end{aligned}
$$

6 More Extensions

6.1 Global shock

With some simple modifications, the models can include a global shock. In the finite model, let θ represent a global shock. By the realization of the global shock, the probability of the realization of $\mathbf{s}_A$ would change. In summary, if we replace $\Pr(\mathbf{s}; A)$ in section 2 with $\Pr(\mathbf{s}; A, \theta)\Pr(\theta)$, the analysis can be carried out in a similar manner. $DG_i(e_i'|A, z_i)$ is defined as

$$DG_i(e_i'|A, z_i) := \left[\sum_{\theta\in\Theta,\mathbf{s}\in\mathbf{S}_A} v_i(z_i^{\mathbf{s}})\Pr(\mathbf{s}; A|e_i', \theta)\Pr(\theta) - E_i(e_i')\right] - \left[\sum_{\theta\in\Theta,\mathbf{s}\in\mathbf{S}_A} v_i(z_i^{\mathbf{s}})\Pr(\mathbf{s}; A, \theta)\Pr(\theta) - E_i(e_i)\right].$$

In the continuum model, similar modifications can be made: $\varphi(s; e_i, e_j)$ is replaced by $\varphi(s; e_i, e_j, \theta)\varphi(\theta)$. Therefore, $DG_i(e_i'|j, e_i, e_j, z_i)$ is defined as

$$DG_i(e_i'|j, e_i, e_j, z_i) := \left[\sum_{\theta,s} v_i(z_i^s)\varphi(s; e_i', e_j, \theta)\varphi(\theta) - E_i(e_i')\right] - \left[\sum_{\theta,s} v_i(z_i^s)\varphi(s; e_i, e_j, \theta)\varphi(\theta) - E_i(e_i)\right].$$

6.2 Correlated Equilibrium of Games inside Firms

In the contracts in section 2 and 3, randomization was only over the consumption vector. But, the extension to a random contract over effort can be modeled as in Rahman (2005a). The incentive compatibility constraints for the finite economy with IC_l are replaced by

$$\sum_{e_i,e_j}\sum_{A\ni(i,j,e_i,e_j)}\sum_{z_i} DG_i(e_i'|A, z_i, z_j)X_{ij}(A, z_i, z_j) \leq 0$$

The incentive compatibility constraints for other economies can be similarly defined.

6.3 Restriction of Non-random Contracts

In the contracts studied in section 2 and 3, randomization over consumption was assumed to be possible. If random contracts are not allowed, the incentive compatibility constraints are

$$\sum_{A\ni(i,j,e_i,e_j)} DG_i(e_i'|A, z_i, z_j)X_{ij}(A, z_i, z_j) \leq 0$$

in the finite economy with IC_g.

Since the randomization has the role of relaxing the incentive compatibility constraints, the efficiency of the economy would be lower when random contracts are not allowed.

However, the complexity of the lottery trade market would be reduced. In the economy in section 2 with IC_g, a contract is a point in set $\mathcal{A} \times \Delta(\mathbb{R}^{L\times|\mathbf{S}|})$. With the non-random contracts assumption, a contract is a point in set $\mathcal{A} \times \mathbb{R}^{L\times|\mathbf{S}|}$. The description of contract is simpler in the economy with non-random contracts.

6.4 Existence of Spot Market: Spot Prices as a randomization device

In section 2, one role of the firm is the commitment technology to certain consumption levels. In other words, by joining a firm individuals are committed to avoiding spot market transactions after the realization of all uncertainties ($A, \mathbf{s}_A$, and the realization of the public randomization device). However, the restriction on spot market trades in contracts is rarely observed in reality. Thus, it is more realistic in certain applications to not assume the commitment technology. Moreover, it turns out that the possibility of spot market trades may simplify the role of the randomization device.

I analyze how the contract could be decentralized in a way that conforms to the spirit of Arrow-Debreu better.

Modification of the Planner's Problem: In order for the assignment of the planner not to be disturbed by the individuals' spot market transactions, the assignment of the planner should guarantee the lack of gains from the trade among individuals. The lack of gains from trade is guaranteed when the gradients of utility functions point the same direction, which is formalized as

$$\langle \nabla v_k(z_k^{\mathbf{s}}), \nabla v_h(z_h^{\mathbf{s}}) \rangle = ||\nabla v_k(z_k^{\mathbf{s}})|| \cdot ||\nabla v_h(z_h^{\mathbf{s}})||, \forall k, h \in I \cup J$$

where $\langle \cdot, \cdot \rangle$ is the inner product and $|| \cdot ||$ is the norm in $\mathbb{R}^L$.

Therefore, the planner's problem has the following additional constraint.

$$\left[\left\langle \nabla v_i(z_i^{\mathbf{s}}), \nabla v_j(z_j^{\mathbf{s}}) \right\rangle - ||\nabla v_i(z_i^{\mathbf{s}})|| \cdot ||\nabla v_j(z_j^{\mathbf{s}})|| \right] X(A, (z_k)) = 0, \forall i, j \in I \cup J$$

The new third dual constraint is

$$\begin{aligned} 0 \quad \geq \quad & \sum_{(i,j,e_i,e_j) \in A} T_{ij}(A, z_i, z_j) + \left[\sum_{(i,j,e_i,e_j) \in A} q_{ij}(\mathbf{s}) - \sum_i z_i^{\mathbf{s}_A} - \sum_j z_j^{\mathbf{s}_A} \right] p(A, (z_i), (z_j), \mathbf{s}) \\ & - \sum_{i,j \in I \cup J} \eta_{i,j}^{\mathbf{s}}(A, (z_k)) \left[\left\langle \nabla v_i(z_i^{\mathbf{s}}), \nabla v_j(z_j^{\mathbf{s}}) \right\rangle - ||\nabla v_i(z_i^{\mathbf{s}})|| \cdot ||\nabla v_j(z_j^{\mathbf{s}})|| \right] \end{aligned}$$

Again, by the fundamental theorem of linear programming, I get

Proposition 10 *At an optimum, if $X(A, (z_k)) > 0$, the following holds.*

$$0 = \sum_{(i,j,e_i,e_j) \in A} T_{ij}(A, z_i, z_j)$$

The analysis is the same except that the insurer now takes the lack of gains from trade into consideration in assessing the insurance premium. The solution of the planner guarantees that nobody gets gains from trade, therefore nobody trades. Even though nobody trades, spot market prices are defined by

$$\phi(\mathbf{s}; A, (z_k)) \propto \nabla v_k(z_k^{\mathbf{s}}), \forall k.$$

Example 6 *The economy has population* $I = \{i\}, J = \{j\}$*. Utility functions are* $u_k(z_{k,1}, z_{k,2}) = (z_{k,1})^{\delta_{k,1}}(z_{k,2})^{\delta_{k,2}}$ *where* $\delta_{i,1}/\delta_{i,2} \neq \delta_{j,1}/\delta_{j,2}$*. The assignment of the planner puts the consumption on the contract curve of the Edgeworth box in order to ensure the lack of gains from trade. Since* $\delta_{i,1}/\delta_{i,2} \neq \delta_{j,1}/\delta_{j,2}$*, the marginal rate of substitution of good 1 and 2 (essentially the spot market price) are different on each point on the contract curve.*

Now suppose the planner's solution has non-degenerate random contracts. After the realization of $\mathbf{s}$*, the contract determines the allocation on the contract curve randomly. Since there is a one-to-one function between the assignment and spot prices, if spot prices are observed before the exercise of the contract, the assumption of the existence/observation of the randomization device can be replaced with that of the observation of spot prices.*

The example gives a sufficient picture as to how spot market prices can deliver enough information for the exercise of contracts. Suppose all the individuals have strictly concave Cobb-Doulgals utility function with different ratios of the parameters as was in the example. The spot market prices are different on all the points on the contract curve. Instead of the public randomization device announcing (z_k) after the realization of A and $\mathbf{s}$, it can announce the spot market price. The spot market price delivers sufficient information to exercise the equilibrium contracts since there is a one-to-one mapping from the contract hyper-plane to set of spot prices in $\mathbb{R}^L$ by the assumption of different ratios of parameters in the utility functions.

Now suppose those parameters are independently drawn from a certain continuous distribution. The probability that any two individuals have the same ratio of parameters is zero in the finite economy. Therefore, I get the following proposition.

Proposition 11 *Generically, the contract can be implemented with the announcement of spot market prices by the insurer.*

Even though the randomization device could be replaced by spot market prices, the interpretation that spot market prices are realized before individuals are endowed, is not realistic. Typically, endowments and utility functions determine prices, rather than prices determine the endowment in economic models.

7 Conclusion

Two contractual matching economies with moral hazard were studied: one with a finite number of individuals and the other with continuums of individuals. Incentive-constrained efficient random assignments of teams, efforts, and consumption were characterized by the planners' problems. By exploiting the duality of linear programming, necessary conditions for incentive-constrained efficient decentralizations were derived.

Contracts in the finite model are functions of economy-wide shocks. The logical limit of the finite model is that contracts have to depend upon the distribution of the realized idiosyncratic shocks, but contracts in the studied continuum model are functions of idiosyncratic shock. This difference was reconciled by the convergence theorem; hence, the continuum model as a limit of the finite model was justified.

In the finite economy, competitive contractual arbitrageurs, competitive insurers pooling risks in the economy, and public randomization devices implementing matching and random contracts were found to be necessary. Insurers needed to know the details of the economy in order to set the right insurance premiums, and the public randomization device needed to accommodate and signal enormous amounts of information for the allocation of resources.

In the continuum, insurers were eliminated. Therefore, undesirable characteristic of detailed knowledge of the economy was eliminated. The elimination of insurers was made possible by the fact that no cross-subsidies by the insurers were necessary.

The public randomization device implementing random contracts were replaced by independent randomization devices owned by individual firms in the continuum model. Therefore, the undesirable centralized information processing technology is eliminated. The replacement of the public randomization device was obtained by making randomness in contracts more involved; hence, it was shown that random contracts were not only for the relaxation of incentive compatibility, but also for the simplicity of contracts in the sense that less information was required in order to exercise contracts.

However, the public randomization device implementing matching of individuals could not be eliminated in the continuum economy. Therefore, in both of the economies, (i) trades of lotteries were necessary for constrained efficiency in general, (ii) efficiency typically required individuals of the same type to obtain different expected utility when assigned to different firms, and (iii) compensating wage differentials which equalized the utilities of same type individuals in different firms were generally incompatible with efficiency.

The essential reason that the randomization device implementing matching was required for efficiency even in the continuum economy is due to the indivisibility nature of teams: a firm can hire only an integer number of individuals.

The implication from the study is the following. In the finite economy, it is well-known that individuals typically do not behave as price-takers since they perceive the possibility of influencing prices. Even when they behave as price takers and trades of lotteries exist, it is less likely that efficiency would be achieved because of the large informational role of insurers and the existence of the public randomization device implementing random contracts. Therefore, I argue that government intervention could be desirable in a small economy. Even in the continuum model, it is shown that the lottery market and the public randomization device implementing random assignment of individuals to teams is required for efficiency. Considering that such a market and the public randomization device are not observed in the real world, a kind of government intervention on the job matching would be desirable when there is no adverse selection problem. However, it is not clear how the result would change when adverse selection problem is introduced to the model.

The other contribution of the paper is the usage of linear programming to model a finite ecconomy. Once a planners problem is set without consideration of decentralized economy, dual linear program of the planners problem characterizes an idealized environment for efficiency of the decentralized economy. Under the characterized environment, the welfare theorems are automatically derived. Therefore, the roles of the definition of equilibrium and the welfare theorems are reversed: welfare theorems become the assumption and the description of equilibrium becomes the theorem.

A Proofs

A.1 Review of Finite Dimensional Linear Programming

Refer to http://www.sscnet.ucla.edu/04S/econ201c-1/LP.pdf

A.2 Infinite Dimensional Linear Programming

The formulation of the finite model is given. The formulation of the continuum model is omitted because the proof is almost identical.

A.2.1 Basic Concepts and Notations

Let Z be a compact set in $\mathbb{R}^{L\times|\mathbf{S}_A|}$

- $\mathbf{C}(Z)$: Banach space of continuous functions on R equipped with the supremum norm; i.e. $\|f\| := \sup_{z\in R}|f(z)|$.
- $\mathbf{M}(Z)$: Banach space of countably additive Borel measures on R equipped with the total variation norm; i.e. $\|\nu\| = \sup_\pi \sum_{E_i} |\nu(E_i)|$ over all finite measureable partitions π of R.

Let $m \in \mathbf{M}(Z)$ be the *Lesbegue* measure. Let R be a compact disc in $\mathbb{R}^{L\times|\mathbf{S}_A|}$, $R = \{z \in \mathbb{R}^{L\times|\mathbf{S}_A|} | \langle z, z\rangle \leq R^{L\times|\mathbf{S}_A|}\}$ with a slight abuse of notation.

- $X_i(A), X_j(A) \in \mathbf{M}(R)$
- $X_{ij}(A) \in \mathbf{M}(R^2)$
- $X(A) \in \mathbf{M}(R^{|I|+|J|})$

For the notational simplicity, we also define $U_i(z_i; A) := \sum_{\mathbf{s}\in\mathbf{S}_A} v_i(z_i^{\mathbf{s}}) \Pr(\mathbf{s}; A) - E_i(e_i)$. Again, with a slight abuse of notation, we define $U_i(z_i, e_i'; A) := \sum_{\mathbf{s}\in\mathbf{S}_A} v_i(z_i^{\mathbf{s}}) \Pr(\mathbf{s}; A|e_i') - E_i(e_i')$ when i is engaged in firm (i, j, e_i, e_j) in matching A and deviating to e_i'.

Let $\pi_{\mathbf{s}}$ be the projection map from R onto $\{z_i^{\mathbf{s}} : |z_i^{\mathbf{s}}| \leq R\}$, and δ_{z_i} be the Direc's delta function at z_i.

$\langle f, x\rangle = \int f dx$ is well-defined with $f \in \mathbf{C}(Z)$ and $x \in \mathbf{M}(Z)$, and $\langle\cdot,\cdot\rangle$ is a bilinear operation. We write $\langle f, x\rangle_E^Z = \int_E f dx$ for Borel set $E \in \mathcal{B}(Z)$. When $E = Z$, we write $\langle f, x\rangle^Z := \langle f, x\rangle_Z^Z$

Definition 9 *The assignment* $((X_i(A)),(X_j(A)),(X_{ij}(A)),X(A))$ *is feasible if*

$$\sum_{A\in\mathcal{A}}\langle\mathscr{I},X_i(A)\rangle^R\le 1,\forall i\in I,\quad \sum_{A\in\mathcal{A}}\langle\mathscr{I},X_j(A)\rangle^R\le 1,\forall j\in J$$
$$X_i(A)(E)=X_{ij}(A)(E,R),\forall i\in I,\forall A\in\mathcal{A},\forall E\in\mathcal{B}(R)$$
$$X_j(A)(E)=X_{ij}(A)(R,E),\forall j\in J,\forall A\in\mathcal{A},\forall E\in\mathcal{B}(R)$$
$$X_{ij}(A)(E_i,E_j)=X(A)(E_i,E_j,R^{|I|+|J|-2}),\forall(i,j)\in A,\forall A\in\mathcal{A},\forall E_i,E_j\in\mathcal{B}(R)$$
$$\left\langle[\sum z_k^{\mathbf{s}}-\sum_{(i,j)\in A}q_{ij}(\mathbf{s})],X(A)\right\rangle_{\times_i E_i}^{R^{|I|+|J|}}\le 0,\forall\mathbf{s}\in\mathbf{S},\forall E_i\in\mathcal{B}(R)$$
$$\sum_{A\ni(i,j,e_i,e_j)}\left\langle\left[U_i(z_i,e_i';A)-U_i(z_i;A)\right],X_{ij}(A)\right\rangle_{E_j}^R\le 0,\forall E_j\in\mathcal{B}(R)$$

Given the definition of feasible allocation, we can state:

Definition 10 *The planner's problem is to find* $(X_i(A),X_j(A),X(i,j,A))$ *to attain*

$$\begin{aligned} g=\quad&\sup\quad \sum_i\sum_A\langle\lambda_iU_i(\cdot;A),X_i(A)\rangle^R+\sum_j\sum_A\langle\lambda_jU_i(\cdot;A),X_j(A)\rangle^R\\ &s.t.\quad (X_i(A),X_j(A),X_{ij}(A),X(A))\ \textit{is feasiible}\end{aligned}$$

A.2.2 Derivation of dual in Bilateral Contractual Matching Market

From the following equality, the result follows.

$$\begin{aligned}\mathcal{L}\ =\ &\sum_i\sum_A\lambda_i\langle U_i(\cdot;A),X_i(A)\rangle+\sum_i\sum_A\lambda_j\langle U_j(\cdot;A),X_j(A)\rangle\\
&+\sum_i y_i\left(1-\sum_A\langle\mathscr{I},X_i(A)\rangle\right)+\sum_j y_j\left(1-\sum_A\langle\mathscr{I},X_j(A)\rangle\right)\\
&-\sum_i\sum_A\left\langle t_i(A),X_i(A)-\langle\mathscr{I},X_{ij}(A)\rangle^R\right\rangle^R+\sum_j\sum_A\left\langle t_j(A),X_j(A)-\langle\mathscr{I},X_{ij}(A)\rangle^R\right\rangle^R\\
&-\sum_{i,j}\sum_{A\ni(i,j)}\left\langle T_{ij}(A),X_{ij}(A)-\langle\mathscr{I},X(A)\rangle^{R^{|I|+|J|-2}}\right\rangle^{R^2}\\
&+\sum_{\mathbf{s}}\sum_A\left\langle p(A,\mathbf{s}),\left[\sum_k z_k-\sum_{(i,j)\in A}q_{ij}(\mathbf{s})\right]X(A)\right\rangle^{R^{|I|+|J|}}\\
&-\sum_i\sum_{j,e_i,e_j}\sum_{A\ni(i,j,e_i,e_j)}\sum_{e_i'}\left\langle\alpha_i(e_i'|A,\cdot),\left\langle DG_i(e_i'|A,\cdot),X_{ij}(A)\right\rangle^R\right\rangle^R\\
&-\sum_j\sum_{i,e_i,e_j}\sum_{A\ni(i,j,e_i,e_j)}\sum_{e_j'}\left\langle\alpha_j(e_j'|A,\cdot),\left\langle DG_j(e_j'|A,\cdot),X_{ij}(A)\right\rangle^R\right\rangle^R\end{aligned}$$

$$
\begin{aligned}
\mathcal{L} \;=\; & \sum_i y_i + \sum_j y_j \\
& + \sum_i \sum_A \langle [\lambda_i U_i(\cdot; A) - y_i - t_i(A)], X_i(A) \rangle^R \\
& + \sum_j \sum_A \langle [\lambda_j U_j(\cdot; A) - y_j - t_j(A)], X_j(A) \rangle^R \\
& + \sum_A \sum_{(i,j,e_i,e_j)\in A} \langle t_i(A) + t_j(A) - T_{ij}(A), X_{ij}(A) \rangle^{R^2} \\
& - \sum_{(i,j,e_i,e_j)} \sum_{e_i'} \left\langle \sum_{A \ni (i,j,e_i,e_j)} \langle \alpha_i(e_i'|j,e_i,e_j), DG_i(e_i'|A) \rangle^R, X_{ij}(A) \right\rangle^{R^2} \\
& - \sum_{(i,j,e_i,e_j)} \sum_{e_j'} \left\langle \sum_{A \ni (i,j,e_i,e_j)} \langle \alpha_j(e_j'|i,e_i,e_j), DG_j(e_j'|A) \rangle^R, X_{ij}(A) \right\rangle^{R^2} \\
& - \sum_A \sum_{\mathbf{s}} \left\langle \sum_{(i,j,e_i,e_j)\in A} T_{ij}(A) - [\sum_k z_k - \sum_{(i,j)\in A} q_{ij}(\mathbf{s})] p(A,\mathbf{s}), X(A) \right\rangle^{R^{|I|+|J|}}
\end{aligned}
$$

∎

A.2.3 Proof of Proposition 3 (Fundamental Theorem of Linear Programming)

Proof is direct from applying Gretsky, Ostroy, and Zame (1992) to the described infinite dimensional linear program.

A.2.4 Finite Support of Allocation: Carathéodory Theorem on Convexification

The space of non-money commodities is finite-dimensional, $\mathcal{E}$ is assumed to be finite, and $\mathbf{S}$ is also finite. Therefore, support of $X(\cdot,\cdot)$ is finite from Carathéodory Theorem on convexification.

A.3 Other Proofs

A.3.1 Proof of Proposition 4 (Complementary Slackness)

The results are direct application of Complementary Slackness of linear programming.

A.3.2 Proof of Lemma 1

Let $X_k(\cdot)$, $X_{ij}(\cdot)$, and $X(A,(z_k))$ be an optimal solution of the planner's problem, by complementary slackness of linear programming,

$$
\left[y_k - \sum_{\mathbf{s}\in\mathbf{S}_A} v_k(z_k^{\mathbf{s}}) \Pr(\mathbf{s}; A) - E(e_k) + t_k(A, z_k) \right] X_k(A, z_k) = 0
$$

Summing it over A and z_k proves the equality for the first line.

From complementary slackness of linear programming,

$$
\left[t_i(A,z_i) + t_j(A,z_j) - T_{ij}(A,z_i,z_j) - \sum_{e_i'} \alpha_i(e_i'|A,z_j) DG_i(e_i'|A,z_i,z_j) - \sum_{e_j'} \alpha_j(e_j'|A,z_i) DG_i(e_i'|A,z_i,z_j) \right] X_{ij}(A,z_i,z_j) = 0
$$

Summing them over (z_i, z_j),

$$\sum_{z_i} t_i(A, z_i)X_i(A, z_i) + \sum_{z_j} t_j(A, z_j)X_j(A, z_j) - \sum_{z_i,z_j} T_{ij}(A, z_i, z_j)X_{ij}(A, z_i, z_j)$$
$$-\sum_{z_j}\sum_{z_i}\sum_{e'_i} \alpha_i(e'_i|A, z_j)DG_i(e'_i|A, z_i, z_j)X_{ij}(A, z_i, z_j) - \sum_{z_i}\sum_{z_j}\sum_{e'_j} \alpha_j(e'_j|A, z_i)DG_i(e'_i|A, z_i, z_j)X_{ij}(A, z_i, z_j) = 0$$

It is left to show the last two terms are zero. The second to the last term can be written

$$\sum_{z_j}\sum_{e'_i} \alpha_i(e'_i|A, z_j) \sum_{z_i} DG_i(e'_i|A, z_i, z_j)X_{ij}(A, z_i, z_j) = 0$$

Again, from Proposition 4, $\alpha_i(e'_i|A, z_j) \sum_{z_i} DG_i(e'_i|A, z_i, z_j)X_{ij}(A, z_i, z_j) = 0$. Therefore, the desired result is shown.

The third statement can be proved in a similar manner.

For the last statement, the result follows directly from Proposition 4 since

$$[\sum_{(i,j)\in A} q_{ij}(\mathbf{s}) - \sum_i z_i^{\mathbf{s}_A} - \sum_j z_j^{\mathbf{s}_A}]p(A, (z_i), (z_j), \mathbf{s})X(A, (z_k)) = 0$$

A.3.3 Proof of Lemma 2

Suppose not, i.e. $\sum_A \sum_{z_k} t_k(A, z_k)X_k(A, z_k) \neq 0$ for some i. Define new variables

$$\begin{aligned}
\hat{y}_k &:= y_k + \sum_{\tilde{A}}\sum_{\tilde{z}_k} t_k(\tilde{A}, \tilde{z}_k)X_k(A, z_k) \\
\hat{t}_k(A, z_k) &:= t_k(A, z_k) - \sum_{\tilde{A}}\sum_{\tilde{z}_k} t_k(\tilde{A}, \tilde{z}_k)X_k(A, z_k) \\
\hat{T}_{ij}(A, z_i, z_j) &:= T_{ij}(A, z_i, z_j) - \sum_{\tilde{A}}\sum_{\tilde{z}_i} t_i(\tilde{A}, \tilde{z}_i)X_i(A, z_i) - \sum_{\tilde{A}}\sum_{\tilde{z}_j} t_j(\tilde{A}, \tilde{z}_j)X_j(A, z_j)
\end{aligned}$$

Then the first and the second dual constraints hold trivially. The following equalities prove that the third constraint holds.

$$\begin{aligned}
& \sum_{(i,j,e_i,e_j)\in A} \hat{T}_{ij}(A, z_i, z_j) = \sum_{(i,j,e_i,e_j)\in A} T_{ij}(A, z_i, z_j) \\
\iff & \sum_k \sum_A \sum_{z_k} t_k(A, z_k)X_k(A, z_k) = 0 \\
\iff & \sum_A \sum_{(z_k)} \sum_{(i,j,e_i,e_j)\in A} T_{ij}(A, z_i, z_j)X(A, (z_k)) = 0 \text{ by lemma 2}
\end{aligned}$$

Therefore, the result follows. ∎

A.3.4 Proof of Lemma 3

From [ARB_IC_g], it is enough to show

$$\sum_{A\ni(i,j,e_i,e_j)} \sum_{z_i,z_j} \left[\sum_{e'_i} \alpha_i(e'_i|A, z_j)DG_i(e'_i|A, z_i) + \sum_{e'_j} \alpha_j(e'_j|A, z_i)DG_j(e'_j|A, z_j)\right] Q_{ij}(A, z_i, z_j) \leq 0,$$

which is true since

$$\left[\sum_{z_i} DG_i(e_i'|A,z_i)Q_{ij}(A,z_i,z_j)\right] \leq 0 \quad \text{and} \quad \left[\sum_{z_j} DG_j(e_j'|A,z_j)Q_{ij}(A,z_i,z_j)\right] \leq 0$$

from the incentive compatibility constraints. ∎

A.3.5 Proof of Lemma 4

From (1), it is enough to show

$$\sum_{A\ni(i,j,e_i,e_j)} \sum_{z_i,z_j} \left[\sum_{e_i'} \alpha_i(e_i'|A)DG_i(e_i'|A,z_i) + \sum_{e_j'} \alpha_j(e_j'|A)DG_j(e_i'|A,z_j)\right] Q_{ij}(A,z_i,z_j) \leq 0,$$

which is true since

$$\left[\sum_{A\ni(i,j,e_i,e_j)} \sum_{z_i,z_j} DG_i(e_i'|A,z_i)Q_{ij}(A,z_i,z_j)\right] \leq 0 \quad \text{and} \quad \left[\sum_{A\ni(i,j,e_i,e_j)} \sum_{z_i,z_j} DG_j(e_j'|A,z_j)Q_{ij}(A,z_i,z_j)\right] \leq 0$$

from the incentive compatibility constraints. ∎

A.3.6 Proof of Lemma 5

Proof. For $Q(A,(z_k))$ such that $X(A,(z_K)) > 0$, it is trivial from proposition 3. For $Q(A,(z_k))$ such that $X(A,(z_k)) = 0$, it is enough to show

$$p(A,\mathbf{s},(z_k)) \left[\sum_{(i,j)\in A} q_{ij}(\mathbf{s}) - \sum_k z_k^{\mathbf{s}}\right] \geq 0,$$

which is trivially true. ∎

A.3.7 Proof of Proposition 6 (Fundamental Theorem of Linear Programming)

Omitted. Refer to the proof of the finite model.

A.3.8 Proof of Proposition 7 (Complementary Slackness)

The results are direct application of Complementary Slackness of linear programming.

A.3.9 Proof of Lemma 6

It can be proved in the same manner as was done in the proof of Lemma 1.

A.3.10 Proof of Lemma 7

The proof is almost identical to that of Shapley (1969). Before getting into the proof, I prove a lemma.

Lemma 9 $y_i \geq 0$

Proof. Replace the resource constraint of the planner by

$$\sum_{i,j}\sum_{e_i,e_j}\sum_{z_i,z_j}\sum_{s}[z_i^s + z_j^s]\varphi(s;e_i,e_j)x_{ij}(e_i,e_j,z_i,z_j) \leq \sum_{i,j}\sum_{e_i,e_j}\sum_{z_i,z_j}\sum_{s} q(s)\varphi(s;e_i,e_j)x_{ij}(e_i,e_j,z_i,z_j)$$

Also let $U_i(x_i(\cdot))$ be

$$U_i(X_i(\cdot,\cdot)) := \sum_{z_i}\sum_{j,e_i,e_j}\left[\sum_{s\in S} v_i(z_i^s)\varphi(s;e_i,e_j) - E_i(e_i)\right] x_i(j,e_i,e_j,z_i).$$

Then $y_i \geq 0$, since the probability constraint is an inequality constraint

$$\sum_{j,e_i,e_j}\sum_{z_i} x_i(j,e_i,e_j,z_i) \leq 1, \forall i \in I \cup J$$

The solution of the new planner's program is trivially in the domain of the original planner's program. The planner's solution of the original program cannot be smaller than that of the new one. Therefore, the value of individual i cannot go down. So $y_i \geq 0$ still holds. ∎

Let $\Gamma(\lambda)$ denote the planner's linear program with weight λ, and $\Gamma^{-1}(\lambda)$ denote the dual linear program. Let $F(\lambda)$ denote the feasible set for $\Gamma(\lambda)$, and let

$$\psi(\lambda) := -\left[\sum_{j,e_i,e_j}\sum_{z_i} t_i(j,e_i,e_j,z_i)x_i(j,e_i,e_j,z_i)\right]_{i\in I\cup J} - p\sum_{z_i} z_i x_i(\emptyset, z_i)$$
$$\text{where} \quad x_i(\cdot) \in \operatorname{argmax}\Gamma(\lambda), t_i(\cdot) \in \operatorname{argmin}\Gamma^{-1}(\lambda)$$

The set of all such vectors, denoted by $P(\lambda)$ is non-empty by proposition 6. If no constraints are redundant in the primal, the relevant dual solution is unique. If there are redundant constraints, the relevant dual value could be arbitrary subject to the sum of the dual values corresponding to the redundant constraints is a constant. Also, $\lambda_i = 0$ implies $\psi_i(\lambda) \geq 0$ from [indv] and $y_i \geq 0$. Therefore, $P(\lambda)$ is convex and compact; hence, it varies upper-semi-continuously with λ. If $P(\lambda)$ contains a zero vector, then $\psi(\lambda)$ is feasible (i.e. $\psi(\lambda) \equiv 0$), and we are through.

Define the set-valued function T by

$$T(\lambda) = \lambda + P(\lambda) = \{\lambda + \pi | \pi \in P(\lambda)\}.$$

Let A be a simplex in the hyperlane $\{\alpha | \sum_{k\in I\cup J}\alpha_k = 1\}$, large enough to contain all sets $T(\lambda)$, $\lambda \in \Lambda = \{\lambda \geq 0 | \sum \lambda_k = 1\}$, as well as Λ itself; the upper-continuity of T makes this possible – i.e., makes $T(\Lambda)$ compact. Extend the definition of T to A by

$$T(\alpha) = T(f(\alpha)), \quad \text{where} \quad f_k(\alpha) = \frac{\max(0,\alpha_k)}{\sum_h \max(0,\alpha_h)}$$

According to Kakutani's theorem, there is a "fixed point" α^* satisfying $\alpha^* \in T(\alpha^*)$. Denote $f(\alpha^*)$ by λ^*. Suppose first that $\alpha^* \neq \lambda^*$. Then $\alpha^* \in A - \Lambda$, and for some i, $\lambda_i^* = 0 > \alpha_i^*$. But $\alpha^* \in T(\lambda^*) = \lambda^* + P(\lambda^*)$, hence $\pi_i^* < 0$ for some $\pi^* \in P(\lambda^*)$. Since $\psi_i(\lambda^*) \geq 0$, the feasible payoff vector $\psi(\lambda^*) - \pi^* \in F(\lambda^*)$ gives player i a positive amount. But this is impossible without side payments, since all his payoffs in $\Gamma(\lambda^*)$ are zero. I conclude that $\alpha^* = \lambda^*$; hence that $0 \in P(\lambda^*)$; hence that

$$\psi(\lambda^*) \in F(\lambda^*)$$

Lemma 7 is shown. ∎

A.3.11 Discussion on Assumption 5

Lemma 7 does not state that there exist $\lambda \gg 0$. Without the existence of such λ, the analysis is not meaningful since

$$y_i \geq \lambda_i \left[\sum_s v_i(z_i)\varphi(s; e_i, e_j) - E_i(e_i)\right] - t_i(j, e_i, e_j, z_i), \quad y_i \geq \lambda_i v_i(z_i) - pz_i$$

describe individuals' behavior. With $\lambda_i = 0$, the inequality does not represent individuals' optimization. For the existence of $\lambda \gg 0$, an assumption is needed. Definition 4.3.5 of a *linked* allocation from Mas-Colell (1985) is the following.

4.3.5. Definition. *The allocation x is linked if:*

1. *for each j, $\partial_h u_j(x_j)x_j^h \neq 0$ for some h;*
2. *for each produced h, $\partial_h u_j(x_j)x_j^h \neq 0$ for some j; and*
3. *the set of agents cannot be partitioned in two groups having no desired commodity in common. More formally, it is not possible to label commodities and agents in such a way that for some $1 < r \leq \ell$, $1 < s \leq r$ we have that $\partial_h u_j(x_j)x_j^h \neq 0$ only if $j \geq s$ and $h \geq r$ or $j < s$ and $h < r$.*

The definition is modified with consideration that (1) 'commodities' in the continuum model is matching/consumption, and that (2) commodities are individualized goods. The first constraint is modified into

1. *for each i, $\sum_{z_i} \left[\sum_s v_i(z_i^s)\varphi(s; e_i, e_j) - E_i(e_i)\right] x_i(j, e_i, e_j, z_i) \neq 0$ for some (j, e_i, e_j).*

The second constraint is modified into

2. *for each (i, j, e_i, e_j) s.t. $\sum_{z_i, z_j} x_{ij}(e_i, e_j, z_i, z_j) > 0$, $\sum_{z_i} \left[\sum_s v_i(z_i^s)\varphi(s; e_i, e_j) - E_i(e_i)\right] x_i(j, e_i, e_j, z_i) \neq 0$.*

The third constraint is meaningless in the context of the contractual matching market since (k, h) cannot consume (i, j, e_i, e_j). The second modified condition implies the first, therefore the definition is modified into the following.

Definition 11 *The allocation $[x_i(\cdot)]_{i \in I \cup J}$ is linked if:*

for each (i, j, e_i, e_j) such that $\sum_{z_i, z_j} x_{ij}(e_i, e_j, z_i, z_j) > 0$, $\sum_{z_i} \left[\sum_s v_i(z_i^s)\varphi(s; e_i, e_j) - E_i(e_i)\right] x_i(j, e_i, e_j, z_i) \neq 0$.

Sufficiency of the condition for the existence of $\lambda \gg 0$ is done in the same way as proposition 4.3.6. was done in Mas-Collel (1985). The commodity space that Mas-Collel considered was of finite dimension, while that of the contractual matching economy was of infinite dimension. However, from Carathéodory's theorem on convex hull, the number of purchased good is finite. Therefore, the proof by induction can be applied in the same way as in Mas-Collel. ∎

A.3.12 Proof of Lemma 8

From inequality [arb], it is enough to show

$$0 \leq \sum_{z_i, z_j} \left[\sum_{e_i'} \alpha_i(e_i'|e_i, e_j, z_j) DG_i(e_i'|e_i, e_j, z_i) + \sum_{e_j'} \alpha_j(e_j'|e_i, e_j, z_i) DG_j(e_j'|e_i, e_j, z_j)\right] \xi_{ij}(e_i, e_j, z_i, z_j)$$

which is trivially true by the incentive compatibility constraints. ∎

A.3.13 Proof of Theorem 3

Suppose the converse, then

$$\lim_{n\to\infty} V^F(n) > V^C,$$

which means that there is a large number N such that $V^F(N) > V^C$.

For given $X(A,(z_k))$, let $i \in N' \subset N$ be the set of individuals who have

$$q_{ij}(\mathbf{s}) = \text{constant, but } z_i^{\mathbf{s}} \neq z_i^{\mathbf{s}'} \text{ for all } \mathbf{s},\mathbf{s}' \in \tilde{\mathbf{S}}_i(\subset \mathbf{S}) \text{ in } X_i(A,z_i)$$

Let us order the elements in $\tilde{\mathbf{S}}_i$ as $\{\mathbf{s}_i^1, \mathbf{s}_i^2, \ldots, \mathbf{s}_i^{m_i}, \ldots, \mathbf{s}_i^{|\tilde{\mathbf{S}}_i|}\} = \tilde{\mathbf{S}}_i$. Also, let $a_i^{m_i} := z_i^{\mathbf{s}_i^{m_i}}$.

Define $\tilde{X}(A, ((z'_k)_{k\notin N'}, (z'_k(\tau))_{k\in N'})$ for $\tau \in \{(m_i)_{i\in N'} | m_i \in \mathbf{S}_i\}$ such that

$$z'_k = z_k \text{ if } k \in N \setminus N'$$

$$z_i'^{\mathbf{t}}(\tau) = z_i{}^{\mathbf{t}} \text{ if } i \in N', \mathbf{t} \notin \mathbf{S}_i$$

$$z_i'^{\mathbf{t}}(\tau) = z_i{}^{\mathbf{s}_i^{m_i}} \text{ if } i \in N', \mathbf{t} \in \mathbf{S}_i$$

$$\tilde{X}(A, (z'_k)_{k\notin N'}, (z'_k(\tau))_{k\in N'}) = \frac{\prod_k \Pr_A^{\mathbf{s}_k^{m_k}}}{\sum_\tau \prod_k \Pr_A^{\mathbf{s}_k^{m_k}}} X(A,(z_k)), \text{ where } \tau = (m_i)_{i\in N'}$$

then

$$\sum_\tau \tilde{X}(A, (z'_k)_{k\notin N'}, (z'_k(\tau))_{k\in N'}) = X(A,(z_k))$$

Definitions of $\tilde{X}_i(A, z'_i(\tau))$ and $\tilde{X}_{ij}(A, z'_i(\tau), z'_j(\tau))$ are followed so that the probability clearing constraints hold for the new variables as was in the simple case. Also as was in the last simple case, the value of the objective function and the incentive compatibility constraints are satisfied with $X(A,(z_k(\tau)))$ with $\tau \in \{(m_i)_{i\in N'} | m_i \in \mathbf{S}_i\}$.

Although the resource constraint is not satisfied in general, the following expected resource constraint holds.

$$\sum_\tau \sum_{\mathbf{s}\in\mathbf{S}} \left[\sum_k z'_k(\tau)^{\mathbf{s}} - \sum_{(i,j)\in A} q_{ij}(\mathbf{s}) \right] \tilde{X}(A, (z'_k(\tau))) \Pr_A^{\mathbf{s}} = 0$$

Now interpret the model as the continuum version of $V^F(N)$. Decompose the continuum economy into continuum of identical infinitesimal sub-economies. For each infinitesimal sub-economy, assign the randomized team structure and consumption by the new assignment $\tilde{X}(\cdot)$ described above. For each sub-economy, the resource constraint does not hold, but by transferring resources across sub-economies, the resource constraint as a whole economy holds. Apparently, the assignment $\tilde{X}(\cdot)$ is not optimal for the continuum economy in general. Moreover, assignment/contracts $\tilde{X}(\cdot)$ depend on idiosyncratic shock only now. Therefore,

$$V^C = V^C(N) \geq V^F(N)$$

where the value of the planner's problem in this continuum economy is $V^C(N)$. Now a contradiction is derived since $V^F(N) > V^C$ had been assumed. ∎

References

[1] Arnott, Richard and Joseph E. Stiglitz (1988), *Randomization with Asymmetric Information*, The RAND Journal of Economics 19 (3), 344 - 362

[2] Bennardo, Alberto and Salvatore Piccolo (2005), *Competition with Endogenous Health Risks*, Working Paper

[3] Bennardo, A and P. A. Chiappori (2001), *Bertrand and Walras equilibria under moral hazard*, Journal of Political Economy 111 (4), 785 - 817

[4] Bisin, Alberto and Piero Gottardi (1999), *Competitive Equilibria with Asymmetric Information*, Journal of Economic Theory, 87, 1 - 48

[5] Cole, Harold L. (1989), *Comment: General Competitive Analysis in An Economy With Private Information*, International Economic Review 30 (1), 249 - 252

[6] Cole, Harold L. and Edward C. Prescott (1997), *Valuation Equilibrium with Clubs*, Journal of Economic Theory 74, 19-39

[7] Dam, Kanişka and David Pèrez-Castrillo (2001), *The Principal-Agent Matching Market*, Unitat de Fonaments de l'Anàlisi Econòmica (UAB) and Institut d'Anàlisi Econòmica (CSIC) working paper

[8] Ellickson, Bryan, Birgit Grodal, Suzanne Scotchmer, and William R. Zame (1999), *Clubs and the Market*, Econometrica 67, 1185 - 1218

[9] Ellickson, Bryan, Birgit Grodal, Suzanne Scotchmer, and William R. Zame (2001), *Clubs and the Market: Large Finite Economies*, Journal of Economic Theory, November 2001, 101(1), 40-77.

[10] Garratt, Rod (1995), *Decentralizing Lottery Allocations in Markets with Indivisible Commodities*, Economic Theory 5, 295 - 313

[11] Gretsky, Neil E., Joseph M. Ostroy, and William R. Zame (1992), *The nonatomic assignment model*, Economic Theory No. 2, 103 - 127

[12] Grossman, Sanford J. and Oliver D. Hart, (1983), *Implicit Contracts under Asymmetric Information*, The Quarterly Journal of Economics 98 (3), 123 - 56

[13] Hellwig, Martin (2005), *Nonlinear Incentive Provision in Walrasian Markets: A Cournot Convergence Approach*, Journal of Economic Theory, 120, 1 - 38

[14] Holmstrom, Bengt (1982), *Moral hazard in teams*, Bell Journal of Economics, 13:324 - 340

[15] Jerez, Belén (2003), *A dual characterization of incentive efficiency*, Journal of Economic Theory 112, 1 - 34

[16] Ju, Biung-Ghi. (2005), *Strategy-Proof Risk Sharing*, Games and Economic Behavior 50, 225 - 254

[17] Magill, Michael and Martine Quinzii (2005), *An Equilibrium Model of Managerial Compensation*, Working Paper

[18] Makowski, Louis and Joseph M. Ostroy (1996), *Linear Programming in General Equilibrium*, Working Paper

[19] Makowski, Louis and Joseph M. Ostroy (2003), *Competitive Contractual Pricing with Transparent Teams*, Working Paper

[20] Mas-Colell, Andreu (1985), *The Theory of General Economic Equilibrium: A differentiable approach*, Cambridge University Press

[21] Mookherjee, Dilip (1984), *Optimal Incentive Schemes with Many Agents*, The Review of Economic Studies, No. 3, 433 - 446

[22] Myerson, Roger B. (1991), *Game Theory: Analysis of Conflict*, Harvard University Press

[23] Ostroy, Joesph M. and Joon Song (2005), *Correlated Equilibrium and Pricing of Public Goods*, Working Paper

[24] Prescott, Edward C. and Robert M. Townsend (1984), *Pareto Optima and Competitive Equilibria with Adverse Selection and Moral Hazard*, Econometrica Vol. 52, No. 1, 21 - 45

[25] Prescott, Edward C. and Karl Shell (2002), *Introduction to Sunspots and Lotteries*, Journal of Economic Theory 107, 1 - 10

[26] Prescott, Edward S. and Robert M. Townsend (2000), *Firms as Clubs in Walrasian Markets with Private Information*, Federal Reserve Bank of Richmond Working Paper 00-8

[27] Rahman, David (2005a), *Contractual Pricing with Incentive Constraints*, Working Paper

[28] Rahman, David (2005b), *Optimum Contracts with Public and Private Monitoring via Duality*, Working Paper

[29] Rothschild, Michael, and Joseph Stiglitz (1976) *Equilibrium in Competitive Insurance Markets: An Essay on the Economics of Imperfect Information*, Quarterly Journal of Economics 90, 629 - 49.

[30] Serfes, Konstantinos (2003), *Risk Sharing vs. Incentives: Contract Design under Two-Sided Heterogeneity*, Working Paper

[31] Shapley, Lloyd S., "Utility Comparison and the Theory of Games", *La Decision*: 251-263, Paris: Editions du Centre National de la Recherche Scientifique. (1969) (Reprinted on pp.307-319 of The Shapley Value (Alvin E. Roth, ed.), Cambridge: Cambridge University Press, 1989)

[32] Tommasi, Mariano and Federico Weinschelbaum (2004), *Principal-Agent Contracts under the Threat of Insurance*, Working Paper

[33] Uhlig, Harald (1996), *A Law of Large Numbers for Large Economies*, Economic Theory 8 (1), 41 - 50

[34] Zame, William R. (2005), *Incentives, Contract and Markets – A General Equilibrium Theory of Firms*, Working Paper