Non-Cooperative Games: Equilibrium Existence

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August 2005

Abstract

This entry in *The New Palgrave Dictionary of Economics*, Second Edition, provides a brief overview of equilibrium existence results for continuous and discontinuous non-cooperative games. *JEL* Classification Number: C7

1. Introduction

Nash equilibrium is *the* central notion of rational behavior in noncooperative game theory.[1] Our purpose here is to discuss various conditions under which a strategic form game possesses at least one Nash equilibrium.

Strategic settings arising in economics are often naturally modeled as games with infinite strategy spaces. For example, models of price and spatial competition (Bertrand (1883), Hotelling (1929)), quantity competition (Cournot (1838)), auctions (Milgrom and Weber (1982)), patent races (Fudenberg et. al. (1983)), etc., typically allow players to choose any one of a continuum of actions. The analytic convenience of the continuum from both an equilibrium characterization and comparative statics point of view is perhaps the central reason for the prevalence and usefulness of infinite-action games. Because of this, our treatment will permit both finite-action and infinite-action games.

Games with possibly infinite strategy spaces can be divided into two categories: those with continuous payoffs and those with discontinuous payoffs. Cournot oligopoly models and Bertrand price-competition models with differentiated products, as well as all finite-action games, are important examples of continuous

[1] See Osborne (2005) and Osborne and Rubinstien (1994) for a discussion of Nash equilibrium, including motivation and interpretation.

games, while Bertrand price-competition with homogeneous products, auctions, and Hotelling spatial competition, are important examples in which payoffs are discontinuous. Equilibrium existence results for both continuous and discontinuous games will be reviewed here. We begin with some notation.

A strategic form game, $G = (S_i, u_i)_{i=1}^N$, consists of a positive finite number, N, of players, and for each player $i \in \{1, ..., N\}$, a non empty set of pure strategies, S_i, and a payoff function $u_i : S \to \mathbb{R}$, where $S = \times_{i=1}^N S_i$. The notation s_{-i} and S_{-i} have their conventional meanings: $s_{-i} = (s_1, ..., s_{i-1}, s_{i+1}, ..., s_N)$ and $S_{-i} = \times_{j \neq i} S_j$. Throughout, we shall assume that each S_i is a subset of some metric space and that if any finite number of sets are each endowed with a topology, then the product of those sets is endowed with the product topology.

2. Continuous Games

2.1. Pure Strategy Nash Equilibria

Pure strategy equilibria are more basic than their mixed strategy counterparts for at least two reasons. First, pure strategies do not require the players to possess preferences over lotteries. Second, mixed strategy equilibrium existence results often follow as corollaries of the pure strategy results. It is therefore natural to consider first the case of pure strategies.

Definition 2.1. *$s^* \in S$ is a* pure strategy Nash equilibrium *of $G = (S_i, u_i)_{i=1}^N$ if for every player i, $u_i(s^*) \geq u_i(s_i, s_{-i}^*)$ for every $s_i \in S_i$.*

An important and very useful result is the following.

Theorem 2.2. *If each S_i is a non empty, compact, convex subset of a metric space, and each $u_i(s_1, ..., s_N)$ is continuous in $(s_1, ..., s_N)$ and quasi-concave in s_i, then $G = (S_i, u_i)_{i=1}^N$ possesses at least one pure strategy Nash equilibrium.*

Proof. For each player i, and each $s_{-i} \in S_{-i}$, let $B_i(s_{-i})$ denote the set of maximizers in S_i of $u_i(\cdot, s_{-i})$. The continuity of u_i and the compactness of S_i ensure that $B_i(s_{-i})$ is non empty and also ensure, given the compactness of S_{-i}, that the correspondence, $B_i : S_{-i} \twoheadrightarrow S_i$ is upper hemicontinuous. The quasiconcavity of u_i in s_i implies that $B_i(s_{-i})$ is convex. Consequently, each B_i is upper hemicontinuous, non empty-valued and convex-valued. All three of these properties are therefore inherited by the correspondence $B : S \twoheadrightarrow S$ defined by $B(s) = \times_{i=1}^N B_i(s_{-i})$ for each $s \in S$. Consequently, we may apply Glicksberg's (1952) fixed point theorem to B and conclude that there exists $\hat{s} \in S$ such that $\hat{s} \in B(\hat{s})$. This $\hat{s}$ is therefore a pure strategy Nash equilibrium. Q.E.D.

Remark 1. *The theorem remains valid when "metric space" is replaced by "locally convex Hausdorff topological vector space."*

Remark 2. *The convexity property of strategy sets and the quasiconcavity of payoffs in own action cannot be dispensed with. For example, strategy sets are not convex in matching pennies, and even though the continuity and compactness assumptions hold there, no pure strategy equilibrium exists. On the other hand, in the two-person zero-sum game in which both players' compact convex pure strategy set is* $[-1, 1]$ *and player 1's payoff function is* $u_1(s_1, s_2) = |s_1 + s_2|$ *, all of the assumptions of Theorem 2.2 hold except the quasiconcavity of* u_1 *in* s_1*. But this is enough to preclude the existence of a pure strategy equilibrium because in any such equilibrium player 2's payoff would have to be zero (given* s_1*, 2 can choose* $s_2 = -s_1$*) and 1's payoff would have to be positive (given* s_2*, 1 can choose* $s_1 \neq -s_2$*).*

Remark 3. *More general results for continuous games can be found in Debreu (1952) and Schafer and Sonnenschein (1975). Existence results for games with strategic complements on lattices can be found in Milgrom and Roberts (1990) and Vives (1990).*

2.2. Mixed Strategy Nash Equilibria

A *mixed strategy* for player i is a probability measure, m_i, over S_i. If S_i is finite, then $m_i(s_i)$ denotes the probability assigned to $s_i \in S_i$ by the mixed strategy m_i, and i's set of mixed strategies is the compact convex subset of Euclidean space $M_i = \{m_i \in [0, 1]^{\# S_i} : \sum_{s_i \in S_i} m_i(s_i) = 1\}$.

In general, we shall not require S_i to be finite. Rather, we shall suppose only that it is a subset of some metric space. In this more general case, a mixed strategy for player i is a (regular, countably-additive) probability measure, m_i, over the Borel subsets of S_i; for any Borel subset A of S_i, $m_i(A)$ denotes the probability assigned to A by the mixed strategy m_i. Player i's set of such mixed strategies, M_i, is then convex. Further, if S_i is compact, the convex set M_i is compact in the weak-$*$ topology.[2]

[2]See, e.g. Billingsley (1968).

Extend $u_i : S \to \mathbb{R}$ to $M = \times_{i=1}^N M_i$ by an expected utility calculation.[3] That is, define $u_i(m_1, ..., m_N) = \int_{S_1} ... \int_{S_N} u_i(s_1, ..., s_N) dm_1 ... dm_N$ for all $m = (m_1, ..., m_N) \in M$.[4] Finally, let $\bar{G} = (M_i, u_i)_{i=1}^N$ denote the *mixed extension* of $G = (S_i, u_i)_{i=1}^N$.

Definition 2.3. *$m^* \in M$ is a* mixed strategy Nash equilibrium of $G = (S_i, u_i)_{i=1}^N$ *if m^* is a pure strategy Nash equilibrium of the mixed extension, $\bar{G}$, of G. That is, if for every player i, $u_i(m^*) \geq u_i(m_i, m_{-i}^*)$ for every $m_i \in M_i$.*

Because $u_i(m_i, m_{-i})$ is linear, and therefore quasi-concave, in $m_i \in M_i$ for each $m_{-i} \in M_{-i}$, and because continuity of $u_i(\cdot)$ on S implies continuity of $u_i(\cdot)$ on M (in the weak-$*$ topology), Theorem 2.2 applied to the mixed extension of G yields the following basic mixed strategy Nash equilibrium existence result

Corollary 2.4. *If each S_i is a non-empty compact subset of a metric space, and each $u_i(s)$ is continuous in $s \in S$, then $G = (S_i, u_i)_{i=1}^N$ possesses at least one mixed strategy Nash equilibrium, $m^* \in M$.*

Remark 4. *Note that Corollary 2.4 does not require $u_i(s_i, s_{-i})$ to be quasiconcave in s_i, nor does it require the S_i to be convex.*

Remark 5. *Corollary 2.4 yields von Neumann's (1928) classic result for two-person zero-sum games as well as Nash's (1950, 1951) seminal result for finite games as special cases. To obtain Nash's result, note that if each S_i is finite, then each u_i is continuous on S in the discrete metric. Hence, the corollary applies and we conclude that every finite game possesses at least one mixed strategy Nash equilibrium.*

Remark 6. *To see how Theorem 2.2 can be applied to obtain the existence of mixed strategy equilibria in Bayesian games, see Milgrom and Weber (1985).*

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[3] Hence, the $u_i(s)$ are assumed to be von Neumann-Morgenstern utilities.

[4] This is an extension because we view S as a subset of M; each $s \in S$ is identified with the $m \in M$ that assigns probability one to s.

3. Discontinuous Games

The basic challenge one must overcome in extending equilibrium existence results from continuous games to discontinuous games is the failure of the best reply correspondence to satisfy the properties required for application of a fixed point theorem. For example, in auction or Bertrand price-competition settings, discontinuities in payoffs sometimes preclude the existence of best replies. The best reply correspondence then fails to be non empty valued and Glicksberg's theorem, for example, cannot be applied.

A natural technique for overcoming such difficulties is to approximate the infinite strategy spaces by a sequence of finer and finer *finite* approximations. Each of the approximating finite games is guaranteed to possesses a mixed strategy equilibrium (by Corollary 2.4) and the resulting sequence of equilibria is guaranteed, by compactness, to possesses at least one limit point. Under appropriate assumptions, the limit point is a Nash equilibrium of the original game. This technique has been cleverly employed in Dasgupta and Maskin's (1986) pioneering work, and also by Simon (1987). However, while this finite approximation technique can yield results on the existence of *mixed* strategy Nash equilibria, it is unable to produce equally general existence results for pure strategy Nash equilibria. The reason, of course, is that the approximating games, being finite, are guaranteed to possess mixed strategy, but not necessarily pure strategy, Nash equilibria. Consequently, the sequence of equilibria, and so also the limit point, cannot be guaranteed to be pure.

One might be tempted to conclude that, unlike the continuous game case where the mixed strategy result is a special case of the pure strategy result, discontinuous games require a separate treatment of pure and mixed strategy equilibria. But such a conclusion would be premature. A connection between pure and mixed strategy equilibrium existence results similar to that for continuous games can be obtained for discontinuous games by considering a different kind of approximation. Rather than approximating the infinite strategy spaces by a sequence of finite approximations, one can instead approximate the discontinuous payoff functions by a sequence of continuous payoff functions. This payoff-approximation technique is employed in Reny (1999), whose main result we now proceed to describe. All of the definitions, notation, and conventions of the previous sections remain in effect. In particular, each S_i is a subset of some metric space.[5]

[5]This is for simplicity of presentation only. The results to follow hold in non metrizable settings as well. See Reny (1999).

3.1. Better-Reply Security

Definition. Player i can *secure* a payoff of $\alpha \in \mathbb{R}$ at $s \in S$ if there exists $\bar{s}_i \in S_i$, such that $u_i(\bar{s}_i, s'_{-i}) \geq \alpha$ for all s'_{-i} close enough to s_{-i}.

Thus, a payoff can be secured by i at s if i has a strategy that guarantees at least that payoff even if the other players deviate slightly from s.

A pair $(s, u) \in S \times \mathbb{R}^N$ is in the closure of the graph of the vector payoff function if $u \in \mathbb{R}^N$ is the limit of the vector of player payoffs for some sequence of strategies converging to s. That is, if $u = \lim_n(u_1(s^n), ..., u_N(s^n))$ for some $s^n \to s$.

Definition. A game $G = (S_i, u_i)_{i=1}^N$ is *better-reply secure* if whenever (s^*, u^*) is in the closure of the graph of its vector payoff function and s^* is not a Nash equilibrium, some player i can secure a payoff strictly above u_i^* at s^*.

All games with continuous payoff functions are better-reply secure. This is because if (s^*, u^*) is in the closure of the graph of the vector payoff function of a continuous game, we must have $u^* = (u_1(s^*), ..., u_N(s^*))$. Also, if s^* is not a Nash equilibrium then some player i has a strategy $\bar{s}_i$ such that $u_i(\bar{s}_i, s^*_{-i}) > u_i(s^*)$, and continuity ensures that this inequality is maintained even if the others deviate slightly from s^*. Consequently, player i can secure a payoff strictly above $u_i^* = u_i(s^*)$.

The import of better-reply security is that it is also satisfied in many discontinuous games. For example, Bertrand's price-competition game, many auction games, and many games of timing are better-reply secure.

3.2. Pure Strategy Nash Equilibria

The following theorem provides a pure strategy Nash equilibrium existence result for discontinuous games.

Theorem 3.1. *(Reny (1999)) If each S_i is a non empty, compact, convex subset of a metric space, and each $u_i(s_1, ..., s_N)$ is quasi-concave in s_i, then $G = (S_i, u_i)_{i=1}^N$ possesses at least one pure strategy Nash equilibrium if in addition G is better-reply secure.*

Remark 7. *Theorem 2.2 is a special case of Theorem 3.1 because every continuous game is better-reply secure.*

Remark 8. *A classic result due to Sion (1958) states that every two-person zero-sum game with compact strategy spaces in which player 1's payoff is upper-semicontinuous and quasi-concave in his own strategy, and lower-semicontinuous and quasi-convex in the opponent's strategy has a value and each player has an optimal pure strategy.*[6] *It is not difficult to show that Sion's result is a special case of Theorem 3.1.*

Remark 9. *A related result that weakens quasi-concavity but adds conditions to the sum of the players' payoffs can be found in Baye, Tian, and Zhou (1993). Dasgupta and Maskin (1986) provide two interesting pure strategy equilibrium existence results, both of which require each player's payoff function to upper semicontinuous in the vector of all players' strategies.*

3.3. Mixed Strategy Nash Equilibria

One easily obtains from Theorem 3.1 a mixed strategy equilibrium existence result (the analogue of Corollary 2.4) by treating each M_i as if it were player i's pure strategy set and by applying the definition of better-reply security to the mixed extension $\bar{G} = (M_i, u_i)$ of G. This observation yields the following useful result.

Corollary 3.2. *(Reny (1999)) If each S_i is a non empty, compact, convex subset of a metric space, then $G = (S_i, u_i)_{i=1}^N$ possesses at least one mixed strategy Nash equilibrium if in addition its mixed extension, $\bar{G} = (M_i, u_i)$, is better-reply secure.*

Remark 10. *Better-reply security of $\bar{G}$ neither implies nor is implied by better-reply security of G.*[7]

Remark 11. *Corollary 2.4 is a special case of Corollary 3.2 because continuity of each $u_i(s)$ in $s \in S$ implies (weak-$*$) continuity of $u_i(m)$ in $m \in M$, which implies that the mixed extension, $\bar{G}$, is better-reply secure.*

Remark 12. *Corollary 3.2 has as special cases the mixed strategy equilibrium existence results of Dasgupta and Maskin (1986), Simon (1987) and Robson (1994).*

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[6] Sion does not actually prove the existence of optimal startegies, but this follows rather easily from his compactness assumptions and his result that the game has a value, i.e. that infsup=supinf.

[7] See Reny (1999) for sufficient conditions for better-reply security.

Remark 13. *Theorem 3.1 can similarly be used to obtain a result on the existence of mixed strategy equilibria in discontinuous Bayesian games by following Milgrom and Weber's (1985) seminal distributional strategy approach. One simply replaces Milgrom and Weber's payoff continuity assumption with the assumption that the Bayesian game is better-reply secure in distributional strategies. An example of this technique is provided in the next section.*

3.4. An Application to Auctions

Auctions are an important class of economic games in which payoffs are discontinuous. Furthermore, when bidders are asymmetric, in general one cannot prove existence of equilibrium by construction, as in the symmetric case. Consequently, an existence theorem applicable to discontinuous games is called for. Let us very briefly sketch how Theorem 3.1 can be applied in this case.

Consider a first-price single-object auction with N bidders. Each bidder i receives a private value $v_i \in [0,1]$ prior to submitting a sealed bid, $b_i \geq 0$. Bidder i's value is drawn independently according to the continuous and positive density f_i. The highest bidder wins the object and pays his bid. Ties are broken randomly and equiprobably. Losers pay nothing.

Because payoffs are not quasiconcave in own bids, one cannot appeal directly to Theorem 3.1 to establish the existence of an equilibrium in pure strategy bidding functions. On the other hand, it is not difficult to show that all mixed strategy equilibria are pure and nondecreasing. Hence, to obtain an existence result for pure strategies, it suffices to show that there is an equilibrium in mixed, or equivalently in distributional, strategies.[8]

Because the set of distributional strategies for each bidder is a non-empty compact convex metric space and each bidder's payoff is linear in his own distributional strategy, Theorem 3.1 can be applied so long as a first-price auction game in distributional strategies is better-reply secure. Better-reply security can be shown to hold by using the fact that payoff discontinuities occur only when there are ties in bids and that bidders can always break a tie in their favor by increasing their bid slightly. Consequently, a Nash equilibrium in distributional strategies exists and, as mentioned above, this equilibrium is pure and nondecreasing.

[8] In this context, a distributional strategy for bidder i is a joint probability distribution over his values and bids with the property that the marginal density over his values is f_i (see Milgrom and Weber (1985)).

3.5. Endogenous Sharing Rules

Discontinuities in payoffs sometimes arise endogenously. For example, consider a political game in which candidates first choose a policy from the interval [0,1] and each voter among a continuum then decides for whom to vote. Voters vote for the candidate whose policy they most prefer and if there is more than one such candidate it is conventional to assume that voters randomize equiprobably over them. The behavior of voters in the second stage can induce discontinuities in the payoffs of the candidates in the first stage since a candidate can discontinuously gain or lose a positive fraction of votes by choosing a policy that, instead of being identical to another candidate's policy is just slightly different from it.

An elegant way to handle such discontinuities is suggested by Simon and Zame (1990). In particular, for the political game example above, they would not insist that voters, when indifferent, randomize equiprobably. Indeed, applying subgame perfection to the two-stage game would permit voters to randomize in any manner whatsoever over those candidates whose policies they most prefer. With this in mind, if s is a joint pure strategy for the N candidates specifying a location for each, let us denote by $U(s)$ the resulting *set* of payoff vectors for the N candidates when all best replies of the voters are considered. If no voter is indifferent, then $U(s)$ contains a single payoff vector. On the other hand, if some voters are indifferent (as would be the case if two or more candidates chose the same location) and $U(s)$ is not a singleton, then distinct payoff vectors in $U(s)$ correspond to different ways the indifferent voters can randomize between the candidates among whom they are indifferent.

The significance of the correspondence $U(\cdot)$ is this. Suppose that we are able to select, for each s, a payoff vector $u(s) \in U(s)$ in such a way that some joint mixed strategy m^* for the N candidates is a Nash equilibrium of the induced policy-choice game between them when their vector payoff function is $u(\cdot)$. Then m^* together with the voter behavior that is implicit in the definition of $u(s)$ for each s, constitutes a subgame perfect equilibrium of the original two-stage game. Thus, solving the original problem with potentially endogenous discontinuities boils down to obtaining an appropriate selection from $U(\cdot)$. Simon and Zame (1990) provide a general result concerning the existence of such selections, which they also refer to as "endogenous sharing rules." This method therefore provides an additional tool for obtaining equilibrium existence when discontinuities are present. Simon and Zame's main result is as follows.

Theorem 3.3. *(Simon and Zame (1990)) Suppose that each S_i is a compact subset of a metric space and that $U : S \twoheadrightarrow \mathbb{R}^N$ is a bounded, upper hemicontinuous, non empty-valued, convex-valued correspondence. Then for each player i, there is a measurable payoff function, $u_i : S \rightarrow \mathbb{R}$, such that $(u_1(s), ..., u_N(s)) \in U(s)$ for every $s \in S$ and such that the game $(S_i, u_i)_{i=1}^N$ possesses at least one mixed strategy Nash equilibrium.*

Remark 14. *Theorem 3.3 applies to the political game example above because for any policy choice s of the N candidates, the resulting set of payoff vectors $U(s)$ is convex, a fact that follows from the presence of a continuum of voters. It can also be shown that, as a correspondence, $U(\cdot)$ is upper hemicontinuous.*

Remark 15. *In the context of Bayesian games, an even more subtle endogenous-sharing-rule result can be found in Jackson, Swinkels, Simon and Zame (2002). This result too can be very helpful in dealing with discontinuous games. Indeed, Jackson and Swinkels (2005) have shown how it can be used to obtain equilibrium existence results in a variety of auction settings, including double auctions.*

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