Uncertain Outcomes in Conflict

Extrinsic Uncertainty

- have assumed that the highest bidder – group that provides the most effort – wins
- not so obvious in the case of voting and even less so for street demonstrations or warfare
- often there is an element of uncertainty about who will win
- extrinsic (exogenous) as opposed to intrinsic (mixing)

Sources of Uncertainty

- if individuals independently draw participation costs the total effort of each group is random: it is the sum of independent random decisions on whether or not to paticipate and so total participation follows a binomial distribution as in Palfrey-Rosenthal} or Levine-Palfrey
- individual draws of participation costs may be correlated: for example, bad weather may raise participation costs for all members in regions where a party is heavily concentrated.
- size of two group may be uncertain – for example Shachar-Nalebuff, Federsen-Sandroni, Coate-Conlin
- in voting: random errors in counting, validation of votes, intervention of courts (2000 U.S. Presidential election between Bush-Gore)
- luck – in warfare, for example

Contest Success Function

denote by $p_k(b_k, b_{-k})$ the probability that group k wins the prize given their own bid b_k and the bid of the other group b_{-k}.

previously assumed $p_k(b_k, b_{-k}) = 1 \text{ if } b_k > b_{-k}$ is discontinuous when the bids are the same

now assume that $p_k(b_k, b_{-k})$ a continuous function

two basic properties.

- increasing in b_k
- one party wins the prize for certain $p_k(b_k, b_{-k}) + p_{-k}(b_{-k}, b_k) = 1$

Remark: as we shall see this can be derived from random turnout, but not all random turnout models have this form – in Shachar-Nalebuff and Coate-Conlin outcome depends on size of groups as well as bids

Tullock Function

in voting used by Herrera-Morelli-Nunari

$$p_k(b_k, b_{-k}) = \frac{b_k^\alpha}{b_k^\alpha + b_{-k}^\alpha}.$$

$\alpha \to \infty$ approaches the ordinary all-pay auction in which the highest bidder has probability 1 of winning.

basic finding for the all-pay auction: there is no pure strategy equilibrium - the groups must mix to make the outcome sufficiently uncertain that neither can be sure of winning

in the continuous case pure strategy may exist and are what is ordinarily studied: for example in the Tullock model if α is sufficiently small then there are pure strategy equilibrium

extrinsic uncertainty replaces intrinsic uncertainty

Group Advantage with Pure Strategies

group objective function

is therefore $P(b_k, b_{-k}) - \eta_k C(b_k/\eta_k)$

Proposition: *In any pure strategy equilibrium* b_k, b_{-k} *(if one exists) if* $C"(\varphi) > 0$ *then* $b_L > b_S$ *and the large group receives strictly greater utility than the small group; if* $b_L \leq \eta_S$ *and* $C"(\varphi < 0)$ *then* $b_S > b_L$ *and the small group receives strictly greater utility than the large group.*

Proof of Proposition

$b_L > \eta_S$ then certainly the large party turns out more than the small party, so assume $b_L \leq \eta_S$.

utility to party k from playing b_{-k} rather than b_k must not yield an improvement in utility

$$P(b_k, b_{-k}) - \eta_k C(b_k/\eta_k) \geq (1/2) - \eta_k C(b_{-k}/\eta_k)$$

or

$$P(b_k, b_{-k}) - (1/2) \geq \eta_k C(b_k/\eta_k) - \eta_k C(b_{-k}/\eta_k).$$

For party $-k$ this reads

$$P(b_{-k}, b_k) - (1/2) \geq \eta_{-k} C(b_{-k}/\eta_{-k}) - \eta_{-k} C(b_k/\eta_{-k})$$

using $P(b_{-k}, b_k) = 1 - P(b_k, b_{-k})$

$(1/2) - P(b_k, b_{-k}) \geq \eta_{-k} C(b_{-k}/\eta_{-k}) - \eta_{-k} C(b_k/\eta_{-k})$ or

$$P(b_k, b_{-k}) - 1/2 \leq \eta_{-k} C(b_k/\eta_{-k}) - \eta_{-k} C(b_{-k}/\eta_{-k})$$

The Combination Inequality

$$\eta_k C(b_k/\eta_k) - \eta_k C(b_{-k}/\eta_k) \le \eta_{-k} C(b_k/\eta_{-k}) - \eta_{-k} C(b_{-k}/\eta_{-k}).$$

choose k so that $b_k \ge b_{-k}$ and both sides non-negative

convex case: for $k = S$ must have

$$\eta_S C(b_k/\eta_S) - \eta_S C(b_{-k}/\eta_S) \le \eta_L C(b_k/\eta_L) - \eta_L C(b_{-k}/\eta_L).$$

differentiate $\eta_k C(b_k/\eta_k) - \eta_k C(b_{-k}/\eta_k)$ with respect to η_k

$$C(b_k/\eta_k) - C(b_{-k}/\eta_k) - ((b_k/\eta_k) C'(b_k/\eta_k) - (b_{-k}/\eta_k) C'(b_{-k}/\eta_k))$$

or

$$C(b_k/\eta_k) - (b_k/\eta_k) C'(b_k/\eta_k) - (C(b_{-k}/\eta_k) - (b_{-k}/\eta_k) C'(b_{-k}/\eta_k))\,.$$

Cost vs. Marginal Cost

differentiate $C(\varphi) - \varphi C'(\varphi)$ with respect to φ

$-\varphi C''(\varphi) < 0.$

hence

$$C(b_k/\eta_k) - (b_k/\eta_k)C'(b_k/\eta_k) - (C(b_{-k}/\eta_k) - (b_{-k}/\eta_k)C'(b_{-k}/\eta_k)) < 0$$

so

$$\eta_L C(b_k/\eta_L) - \eta_L C(b_{-k}/\eta_L) < \eta_S C(b_k/\eta_S) - \eta_S C(b_{-k}/\eta_S)$$

a contradiction, so we conclude that $k = L$, that is, $b_L > b_S$.

with $b_L > b_S$ if L lowers bid to b_S had ½ chance of winning, at least the probability of the small group, plus a lower cost, so earns more

for concave case, reverse the role of the two parties

Interior Pure Strategy Equilibrium in the Tullock Model

back to the workhorse model of chore or duty with constant marginal cost

now with Tullock contest success function

$$p_k(b_k, b_{-k}) = \frac{b_k^\alpha}{b_k^\alpha + b_{-k}^\alpha}.$$

interior pure strategy equilibria are what they sound like: Nash equilibrium with $\eta_k \underline{\varphi} < B_k < \eta_k$.

do not exist for all values of the parameters

Equilibrium Strategies

Theorem: *If there is an interior pure strategy equilibrium it is unique and each group choose the common bid* $\hat{b}_k = \alpha V/4$ *and consequently has an equal chance of winning. In the case of a chore the small group is utility advantaged and in the case of a duty the large group is utility advantaged. The utility advantaged group receives a utility advantage of* $(\eta_L - \eta_S)(F + \underline{\varphi})$*. The other group, however, receives a positive level of utility equal to* $(\frac{1}{2} - \frac{1}{4}\alpha)V - F\eta_L + \eta_S\underline{\varphi}$*.*

the utility difference is the same as in the all pay (interior) case

main difference: disadvantaged group does not get zero

Tullock yields higher total surplus

- uncertainty surrounding outcome reduces equilibrium effort provision so lowers costs
- calls to eliminate the electoral college in the U.S. might be misguided.

Why do they bid the same?

objective

$$\frac{b_k^{\alpha}}{b_k^{\alpha}+b_{-k}^{\alpha}}V - \eta_k \left(q_k F + (b_k/\eta_k) - \underline{\varphi}\right).$$

FOC

$$\left(\frac{\alpha b_k^{\alpha} b_{-k}^{\alpha}}{\left(b_k^{\alpha}+b_{-k}^{\alpha}\right)^2}\right) V b_k^{-1} - 1$$

that should do it...

Existence of Interior Pure Strategy Equilibria

Theorem: *In the case of a duty, where $F = 0$, an interior pure strategy equilibrium exists if and only if $(4/V)\eta_L \underline{\varphi} \leq \alpha \leq (4/V)\eta_S$ and either $\alpha \leq 1$ or for both groups*

$$\left(\frac{1}{2} - \frac{1}{4}\alpha - \frac{1}{1 + \left(\frac{1}{4}\alpha(V/\eta_k)\right)^{\alpha}} \right) V + \eta_k \underline{\varphi} \geq 0.$$

note that this definitely needs $\alpha < 4$

Theorem: *In the case of a chore $F > 0$ an interior pure strategy equilibrium exists if and only if $\alpha \leq 2$ and $4F\eta_L/(2-\alpha) \leq V \leq 4\eta_S/\alpha$*

Power Sharing, Efficiency and Federalism

alternative interpretation of the conflict resolution function: not probability of winning, but share of prize (Herrera, Morelli, Nunari)

in a federal system each region would be separately governed: an election would determine how many districts each group controls.

one model of power sharing is the Tullock model with $\alpha = 1$.

just the case of a duty

define W to be the difference between the surplus in the Tullock $\alpha = 1$ interior equilibrium and the auction

$V \geq 4\eta_L \underline{\varphi}$ interiority in both models

increase the stakes up to $V = c\eta_S(1-\underline{\varphi})$ continue to have interiority for both models and $W = V/2 + 2\eta_S\underline{\varphi}$. federalism better, and higher stakes even better

above $V = \eta_S(1-\underline{\varphi})$ up until $V = 4\eta_S$ we remain in the interior for the Tullock model but enter the constrained case for the auction

when $V = 4\eta_S$ Tullock bids approach η_S

in the auction never exceed this and substantial probability they are below it so for higher stakes federalism leads to a welfare loss.

Sources of Uncertainty

uncertainty about how many adherents each group has

$\tilde{\eta}_S$ drawn from a probability distribution with mean η_S

$\tilde{\eta}_L = 1 - \tilde{\eta}_S$.

conflict resolution function is then determined by

$$p_k(b_k, b_{-k}) = \Pr\{b_k\tilde{\eta}_k/\eta_k > b_{-k}\tilde{\eta}_{-k}/\eta_{-k}\}.$$

Nalebuff/Shachar normal distribution (does not respect boundaries)
Coate/Conlin assume beta (uniform a special case)

$$p_k(b_k, b_{-k}) = \Pr\{\widetilde{\eta}_k > \frac{b_{-k}}{b_k(\eta_{-k}/\eta_k)+b_{-k}}\}$$

note that in general this depends on the relative size of the groups as well as the bids

common uniform, reduces to Tullock with parameter 1

Shock to the Objective Function

Herrera-Levine-Martinelli

negatively correlated shock to the objective function of the two groups:

here in the cost rather than the value of the prize

weather for example

Correlated Costs

index types by z_k drawn from continuous strictly increasing cumulative distribution function $G_k(z)$ on $[0, \infty)$.

so $y_k = G_k(z_k)$.

party chooses a type threshold ζ_k.

population is large so that the idiosyncratic component of the shock does not matter

positive parameter $0 < \alpha$.

costs are sufficiently high relative to the prize that $W_k < \eta_k \alpha/(1+\alpha)$.

iid draw u_i from a uniform distribution on $[0, 1]$.

single independent common draw ν from uniform on $[0, 1]$.

set $\nu_S = \nu^{1/\alpha} \text{ and } \nu_L = (1-\nu)^{1/\alpha}$

type is $z_{ik} = \alpha u_i/(1+\alpha)\nu_k$.

Turnout Conditional on Common Shock

$\Pr(z_k \leq \zeta_k|\nu_k) = \Pr(u_i \leq ((1+\alpha)/\alpha)\zeta_k\nu_k|\nu_k)$.

For $\zeta_k \leq \alpha/(1+\alpha)$ this is $\Pr(z_k \leq \zeta_k|\nu_k) = ((1+\alpha)/\alpha)\zeta_k\nu_k$ (since the RHS is no greater than 1)

Observe that $\Pr(z_k \leq \zeta_k) = \int((1+\alpha)/\alpha)\zeta_k\nu^{1/\alpha}d\nu = \zeta_k$ from which we can conclude that for $\zeta_k \leq \alpha/(1+\alpha)$ we have $y_k = z_k$.

cannot be optimal to choose $b_k > W_k$ and $W_k \leq \eta_k\alpha/(1+\alpha)$ for $b_k \leq W_k$ turnout conditional on the common shock ν_k is $((1+\alpha)/\alpha)b_k\nu_k$.

Contest Success

wins if $\eta_k \varphi_k \nu_k > \eta_{-k} \varphi_{-k} \nu_{-k}$ or

$\log(b/(\eta_{-k}\varphi_{-k})) + (1/\alpha)(\log(\nu) - \log(1-\nu) > 0$.

for a uniform ν on $[0,1]$ the random variable $\log(\nu) - \log(1-\nu)$ follows a logistic distribution: so probability of winning is Tullock

$$\frac{b_k^\alpha}{b_k^\alpha + b_{-k}^\alpha}.$$

Exogenous versus Endogenous Uncertainty

equilibria with respect to conflict resolution functions indexed by α.

α increases there is less exogenous uncertainty

$p_k(b_k, b_{-k}, \alpha)$ converge to the all-pay auction as $\alpha \to \infty$ if for all $\epsilon > 0$ we have $p_k(b_k, b_{-k}, \alpha) \to 1$ uniformly on $b_k \geq b_{-k} + \epsilon$.

Facts:

1. equilibria existence

2. they approach the unique equilibrium of the all-pay auction

so there must be uncertainty either exogenous or endogenous

Note: in a large population where $p_k(b_k, b_{-k}, \alpha)$ is due to sampling error, from the law of large numbers we have approximately the all-pay auction result

Upper Hemi-Continuity of the Equilibrium Correspondence

limit of equilibria is an equilibrium

strategies are cumulative distribution functions G_k over bids and the objective function of group k is $U_k(G_k, G_{-k}, \alpha)$ where $\alpha = \infty$ corresponds to the all-pay auction

imagine that G_k is in a compact subset of a finite dimensional space and that U_k is continuous

$\hat{G}_k(\alpha)$ are equilibria for finite α.

compactness implies a limit point $\hat{G}_k(\infty)$

can choose subsequence for which $\lim_{\alpha \to \infty} \hat{G}_k(\alpha) = \hat{G}_k(\infty)$.

are the $\hat{G}_k(\infty)$ equilibria of the all-pay auction?

Implications of Continuity

suppose limits $\hat{G}_k(\infty)$ not an equilibrium

so one group has deviation G_k with

$U_k(G_k, \hat{G}_{-k}(\infty), \infty) > U_k(\hat{G}_k(\infty), \hat{G}_{-k}(\infty), \infty)$.

continuity implies $\lim_{\alpha \to \infty} U_k(G_k, \hat{G}_{-k}(\alpha), \alpha) = U_k(G_k, \hat{G}_{-k}(\infty), \infty)$ and $\lim_{\alpha \to \infty} U_k(\hat{G}_k(\alpha), \hat{G}_{-k}(\alpha), \alpha) = U_k(\hat{G}_k(\infty), \hat{G}_{-k}(\infty), \infty)$ so the strict inequality must hold before the limit is reached.

contradicts hypothesis that $\hat{G}_k(\alpha)$ are equilibria for finite α.

Technical Complications

- strategies are not in a finite dimensional space
- continuity is tricky because the continuous conflict resolution functions are continuous for finite α but converge to a the all-pay auction for which the conflict resolution function is discontinuous.

The Weak Topology

need definition of convergence for probability measures

called the weak topology in the probability theory literature and the weak* topology in the literature on functional analysis.

several equivalent definitions or characterizations of convergence $G_k^\alpha \to G_k^\infty$.

- for any continuous random variable the expectation with respect to G_k^α converges to the expectation with respect to G_k^∞.
- for any open set of bids $\mathcal{B}$ the probability $\Pr(\mathcal{B}|\alpha)$ has limit values that are not smaller than the limit probability. (probability can escape to the boundary of an open set), but probability cannot enter an open set.
- For any closed set of bids $\mathcal{B}$ the probability $\Pr(\mathcal{B}|\alpha)$ has limit values that are not larger than the limit probability. (limit probabilities remain trapped within a closed set

Key Properties of the Weak Topology

the space of probability measures is compact in the weak topology.

for finite α the utility function in the case of a duty is

$U_k(G_k, G_{-k}, \alpha) =$

$\int_0^1 \left[p_k(b_k, b_{-k}, \alpha)V - \eta_k \max\{0, b_k/\eta_k - \underline{\varphi}\}\right] dG_k(b_k)dG_{-k}(b_{-k})$

the expectation of a continuous random variable hence by one of the equivalent definitions of weak convergence must converge whenever the G_k do so weakly.

Existence of Nash Equilibrium

for finite α.

$U_k(G_k, G_{-k}, \alpha)$ is continuous and since it is also concave in G_k (linear in fact) the set of best responses to G_{-k} is convex-valued and upper hemi-continuous.

- Cannot wave hands and mutter: Kakutani fixed point theorem
- Instead wave hands and mutter: Glicksberg fixed point theorem

little practical information what these equilibria are like

probability distributions as having continuous parts given by a density function along and discrete part corresponding to atoms.

Unfortunately there can also be “singular” parts corresponding to Cantor functions - functions which are continuous, increasing, climb from 0 to 1 are differentiable almost everywhere – and yet the derivative is always equal to zero.

Convergence

what happens to $U_k(G_k, G_{-k}, \alpha)$ as $\alpha \to \infty$.

must know that the equilibrium utility converges
$\lim_{\alpha\to\infty} U_k(\hat{G}_k(\alpha), \hat{G}_{-k}(\alpha), \alpha) = U_k(\hat{G}_k(\infty), \hat{G}_{-k}(\infty), \infty)$,

and the the utility from a deviation converges
$\lim_{\alpha\to\infty} U_k(G_k, \hat{G}_{-k}(\alpha), \alpha) = U_k(G_k, \hat{G}_{-k}(\infty), \infty)$.

analysis of deviations can be limited to pure strategy deviations, so need only know that

$$\lim_{\alpha\to\infty} U_k(b_k, \hat{G}_{-k}(\alpha), \alpha) = U_k(b_k, \hat{G}_{-k}(\infty), \infty)$$

Sketch of a Nasty Proof

the problem involves ties. The idea is to replace the all-pay no ties result with an α large low probability of ties result

divide up the space of bids into the set where $|b_k - b_{-k}| \geq \epsilon^2$ and the "diagonal" set $|b_k - b_{-k}| < \epsilon^2$

off-diagonal things are fine: p_k is converging uniformly to a continuous function, so all the U_k's converge nicely.

For the diagonal find a uniform bound $\underline{\Pi}$ such that over intervals of length ϵ parties place probability no more than $\underline{\Pi}\epsilon$ - a nearly continuous density

reason: if they try to lump too much weight on a short interval their opponent would have an incentive to "jump over them"

implication: squares along the diagonal of width and height ϵ have probability of no more than $\underline{\Pi}\epsilon^2$ and as there are $1/\epsilon$ such squares the probability of the diagonal is only $\underline{\Pi}\epsilon$.

An Auction with Exogenous Uncertainty

combine the Tullock model with $\alpha = 1$ and the all-pay auction model by assuming that with with some fixed probability p_0 the outcome is decided by the Tullock model with the remaining probability $1 - p_0$ the outcome is decided on the basis of greatest effort - the all-pay model.

Actually a slight variant on the Tullock model.

Tullock model with $\alpha = 1$

$$p_k(b_k, b_{-k}) = \frac{b_k}{b_k + b_{-k}}$$

or

$$p_k(b_k, b_{-k}) - p_{-k}(b_{-k}, b_k) = \frac{b_k - b_{-k}}{b_k + b_{-k}}$$

instead differential divided by possible votes

$$p_k(b_k, b_{-k}) - p_{-k}(b_{-k}, b_k) = b_k - b_{-k}$$

The Linear Conflict Resolution Function

$$p_k(b_k, b_{-k}) - p_{-k}(b_{-k}, b_k) = b_k - b_{-k}$$

comes from the linear conflict resolution function

$$p_k(b_k, b_{-k}) = \tfrac{1+b_k-b_{-k}}{2}.$$

what is this like?

$$\tfrac{1+b_k-b_{-k}}{2}V - (b_k - \eta_k \underline{y})$$

bang-bang

$V < 2$ only committed voters turn outcomes

$V > 2$ everyone turns outcomes

might give you pause about Tullock...

A Random Turnout Model

fixed fraction of voters $0 \leq \iota \leq 1$ are independents drawn randomly from the two parties (do not change cost distribution)

fraction of voters lost to the independents for each party is $1 - \iota$ and the total loss of voters is proportional to the size of the party

size of a party is given by $(1 - \iota)\eta_k$.

if you intend to bid $b_k = \eta_k \varphi_k$ then taking account of the independents

the actual bid is $(1 - \iota)b_k = (1 - \iota)\eta_k \varphi_k$.

fraction of independent voters that support party k is u_k uniform on $[0, 1]$.

given bids b_k the votes of party k are $(1 - \iota)b_k + \iota u_k$.

Contest Success Function

probability that party k wins is the probability that

$$(1-\iota)b_k + \iota(1-u_{-k}) = (1-\iota)b_k + \iota u_k \geq (1-\iota)b_{-k} + \iota u_{-k}$$

or

$$u_{-k} \leq \frac{\iota+(1-\iota)(b_k - b_{-k})}{2\iota}$$

which is to say

$$\max\{0, \min\{1, \frac{\iota+(1-\iota)(b_k - b_{-k})}{2\iota}\}\}$$

or if $\iota \geq 1/2$

$$\frac{\iota+(1-\iota)(b_k - b_{-k})}{2\iota}.$$

case above is $\iota = 1/2$. $\iota > 1/2$ is fine, $\iota < 1/2$ intractable

$\iota \geq 1/2$ like $\alpha \leq 2$ in Tullock and $\iota < 1/2$ is like $\alpha > 2$.

To Cut to the Chase

- with probability $1-p_0$ the election decided by the greatest effort
- with probability p_0 the election is decided by the linear conflict resolution model, that is by the vote differential.

opponent bidding schedule is G_{-k} let $\tilde{G}_{-k}$ denote the probability of winning schedule derived from G_{-k} and the tie-breaking rule

same as G_{-k} at points of continuity of G_{-k} and in general lies between the left and right limit of G_{-k} inclusive where the value in that range is determined by the tie-breaking rule.

Group Objective Function

$$\left((1-p_0)\,\widetilde{G}(b_k|b_{-k}) + p_0\,\tfrac{1+b_k-b_{-k}}{2}\right)V$$

$$-(1-p_0/2)\max\{0, b_k - \eta_k \underline{y}\}$$

(size of party, hence costs, reduced by number of independents)

or for $b_k \geq \eta_k \underline{\varphi}$

$$\widetilde{G}(b_k|b_{-k})V(1-p_0) - (1-(1-V)p_0/2)\,(b_k - \eta_k \underline{y})$$

$$+(1-p_0/2)\,\eta_k \underline{y} - (1-(1+V)p_0/2)\,\eta_k \underline{y} + p_0\,\tfrac{1-b_{-k}}{2}V$$

the constant term

$$+(1-p_0/2)\,\eta_k \underline{y} - (1-(1-V)p_0/2)\,\eta_k \underline{y} + p_0\,\tfrac{1-b_{-k}}{2}V$$

matters for computing the probability of winning but is irrelevant for decision making

The All Pay Auction

$$\widetilde{G}(b_k|b_{-k})V(1-p_0) - (1-(1+V)p_0/2)\,(b_k - \eta_k \underline{y})$$

is the objective function for an all-pay auction with a prize of size $V(1-p_0)$ and marginal cost of effort $1-(1+V)p_0/2$ [note that this is the same for both parties]

differences with standard all-pay

- the small party gets positive utility
- if marginal cost is negative then both parties turn out all members (tipping with high stakes)
- both parties always have a positive probability of winning

Final Remarks

conflict resolution function/contest success probability

ambiguous as to turnout

does not say whether the uncertainty in the outcome is due to random turnout (it could be) or due to other random events (court intervention).

however some random events (court intervention) may change measured turnout