Repeated Games

Long-Run versus Short-Run Player

a fixed simultaneous move *stage game*

Player 1 is long-run with discount factor δ

actions $a^1 \in A^1$ a finite set

utility $u^1(a^1, a^2)$

Player 2 is short-run with discount factor 0

actions $a^2 \in A^2$ a finite set

utility $u^2(a^1, a^2)$

the "short-run" player may be viewed as a kind of "representative" of many "small" long-run players

Repeated Game

history $h_t = (a_1, a_2, \ldots, a_t)$

null history h_0

behavior strategies $\alpha_t^i = \sigma^i(h_{t-1})$

Equilibrium

Nash: usual definition

Subgame perfect: usual definition, Nash after each history

Observation: the repeated static equilibrium of the stage game is a subgame perfect equilibrium of the finitely or infinitely repeated game

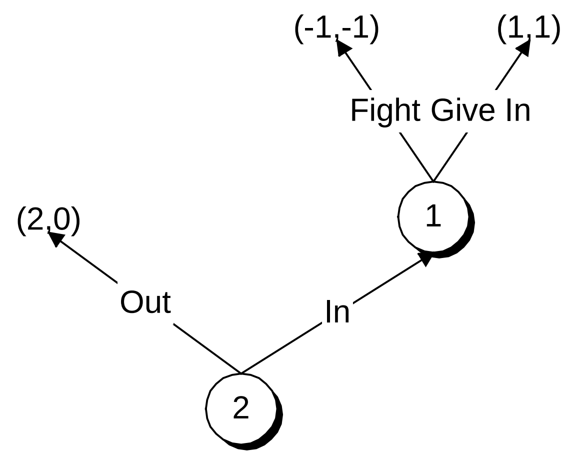
strategies: play the static equilibrium strategy no matter what

“perfect equilibrium with public randomization”

may use a public randomization device at the beginning of each period to pick an equilibrium

key implication: set of equilibrium payoffs is convex

Example: chain store game



normal form

	out	in
fight	2,0*	-1,-1
give in	2,0	1,1**

Nash

subgame perfect is In,Give In

variation on chain store

	out	in
fight	2-ε, 0	-1,-1
give in	2,0	1,1**

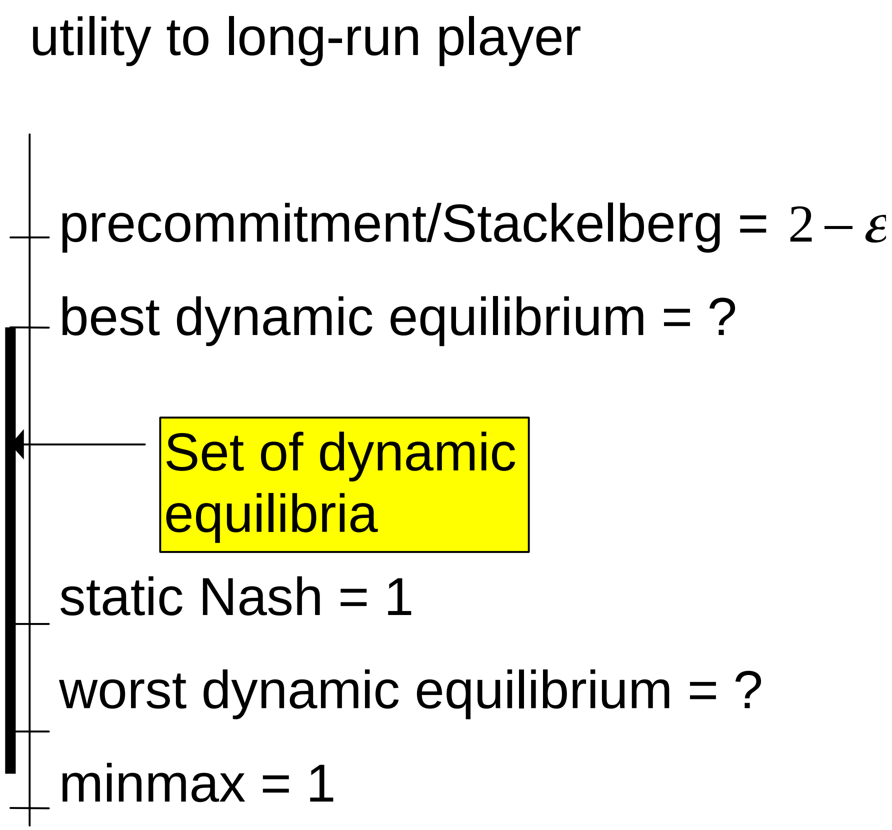
now the only equilibrium is In, Give In

payoff at static Nash equilibrium to LR player: 1

precommitment or Stackelberg equilibrium

precommit to fight get $2-\varepsilon$

minmax payoff to LR player: 1 by giving in


utility to long-run player
precommitment/Stackelberg = $2-\varepsilon$
best dynamic equilibrium = ?
Set of dynamic equilibria
static Nash = 1
worst dynamic equilibrium = ?
minmax = 1

Repeated Chain Store

finitely repeated game

final period: In, Give, so in every period

Do you believe this??

Infinitely repeated game

begin by playing Out, Fight

if Fight has been played in every previous period then play Out, Fight

if Fight was not played in a previous period play In, Give In (reversion to static Nash)

claim: this is subgame perfect

clearly a Nash equilibrium following a history with Give In

SR play is clearly optimal

for LR player

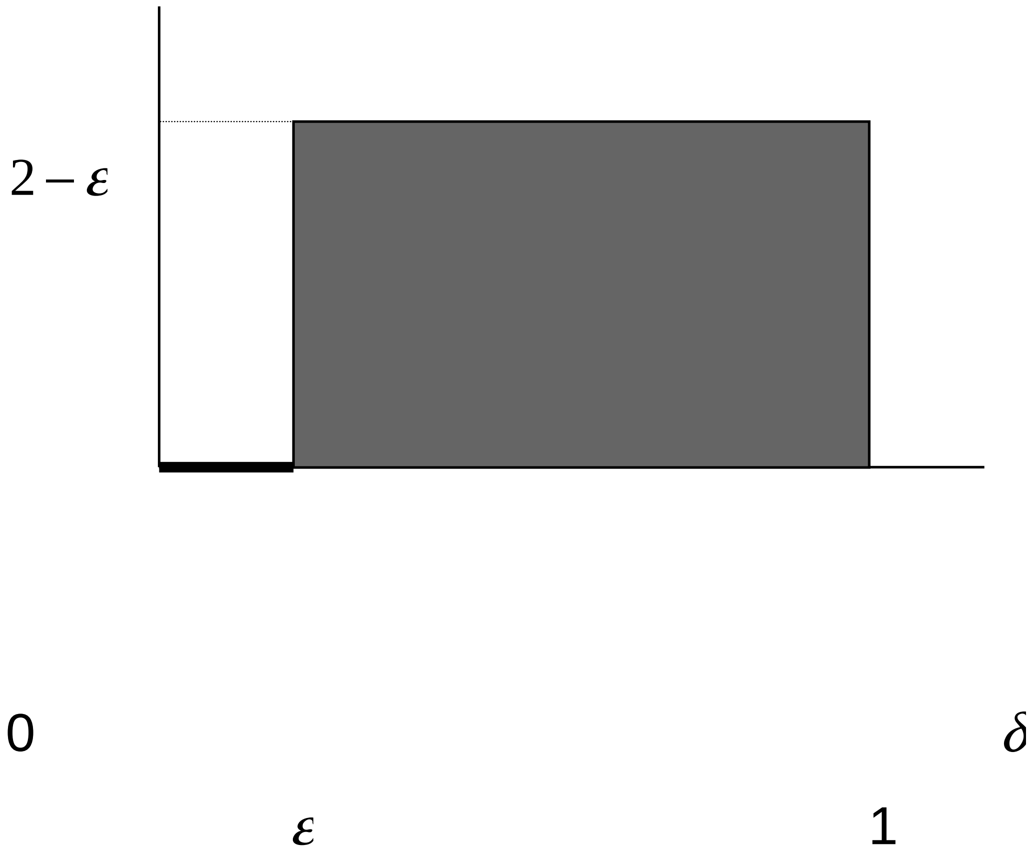
may Fight and get $2-\varepsilon$

or give in and get $(1-\delta)2+\delta 1$

so condition for subgame perfection

$$2-\varepsilon \geq (1-\delta)2+\delta 1$$
$$\delta \geq \varepsilon$$

equilibrium utility for LR



General Deterministic Case

Fudenberg, Kreps and Maskin [1990]

utility to long-run player

max $u^1(a)$

mixed precommitment/Stackelberg

pure precommitment/Stackelberg

$\bar{v}^1$ best dynamic equilibrium

Set of dynamic equilibria

static Nash

$\underline{v}^1$ worst dynamic equilibrium

minmax

min $u^1(a)$

Characterization of Equilibrium Payoff

$\alpha = (\alpha^1, \alpha^2)$ where α^2 is a b.r. to α^1

α represent play in the first period of the equilibrium

$w^1(a^1)$ represents the equilibrium payoff beginning in the next period

$$v^1 \geq (1-\delta)u^1(a^1, \alpha^2) + \delta w^1(a^1)$$

$$v^1 = (1-\delta)u^1(a^1, \alpha^2) + \delta w^1(a^1), \alpha^1(a^1) > 0$$

$$\underline{v}^1 \leq w^1(a^1) \leq \overline{v}^1$$

Characterization of Best/Worst Equilibrium Payoffs

maximize $\bar{v}^1$, minimize $\underline{v}^1$ subject to

$\alpha = (\alpha^1, \alpha^2)$ where α^2 is a b.r. to α^1

$$\bar{v}^1 \geq (1-\delta)u^1(a^1,\alpha^2) + \delta w^1(a^1)$$

$$\bar{v}^1 = (1-\delta)u^1(a^1,\alpha^2) + \delta w^1(a^1), \alpha^1(a^1) > 0$$

$$\underline{v}^1 \geq (1-\delta)u^1(a^1,\alpha^2) + \delta w^1(a^1)$$

$$\underline{v}^1 = (1-\delta)u^1(a^1,\alpha^2) + \delta w^1(a^1), \alpha^1(a^1) > 0$$

$$\underline{v}^1 \leq w^1(a^1) \leq \bar{v}^1$$

Remarks

1) problem simplifies if static Nash = minmax

2) if $v^1 \geq (1-\delta)u^1(a^1,\alpha^2)+\delta w^1(a^1)$ then $v^1 \geq (1-\delta)u^1(a^1,\alpha^2)+\delta \underline{v}^1$

simplification: split into two problems by defining n^1 as static Nash payoff

$$n^1 \leq w^1(a^1) \leq \overline{v}^1$$

$$\underline{v}^1 \leq w^1(a^1) \leq n^1$$

as $\delta \to 1$ $w^1(a^1) \to \overline{v}^1, \underline{v}^1$ in the two problems so this is OK *max problem*

fix $\alpha = (\alpha^1, \alpha^2)$ where α^2 is a b.r. to α^1

$$\bar{v}^1 \geq (1-\delta)u^1(a^1, \alpha^2) + \delta w^1(a^1)$$

$$\bar{v}^1 = (1-\delta)u^1(a^1, \alpha^2) + \delta w^1(a^1), \alpha^1(a^1) > 0$$

$$n^1 \leq w^1(a^1) \leq \bar{v}^1$$

how big can $w^1(a^1)$ be in = case?

Biggest when $u^1(a^1, \alpha^1)$ is smallest, in which case

$$w^1(a^1) = \bar{v}^1$$

$$\bar{v}^1 = (1-\delta)u^1(a^1, \alpha^2) + \delta \bar{v}^1$$

conclusion for fixed α

$$\min_{a^1|\alpha(a^1)>0} u^1(a^1,\alpha^2)$$

i.e. worst in support

$$\bar{v}^1 = \max_{\alpha^2 \in BR^2(\alpha^1)} \min_{a^1|\alpha(a^1)>0} u^1(a^1,\alpha^2)$$

observe:

mixed precommitment $\geq \bar{v}^1 \geq$ pure precommitment

Modified Chain Store Example

	out	in
fight	$2-\varepsilon$, 0	-1,-1
give in	2,0	1,1

p(fight)	BR	worst in support
1	out	$2-\varepsilon$
½<*p*<1	out	$2-\varepsilon$
0<*p*<½	in	-1
p=0	in	1

check: $w^1(a^1) = \dfrac{\overline{v}^1 - (1-\delta)u^1(a^1, \alpha^2)}{\delta} \geq n^1$

as $\delta \to 1$ then $w^1(a^1) \to \overline{v}^1 \geq n^1$

min problem

fix $\alpha = (\alpha^1, \alpha^2)$ where α^2 is a b.r. to α^1

$$\underline{v}^1 \geq (1-\delta)u^1(a^1, \alpha^2) + \delta w^1(a^1)$$

$$\underline{v}^1 \leq w^1(a^1) \leq n^1$$

Biggest $u^1(a^1, \alpha^1)$ must have smallest $w^1(a^1) = \underline{v}^1$

$$\underline{v}^1 = (1-\delta)u^1(a^1, \alpha^2) + \delta \underline{v}^1$$

conclusion

$$\underline{v}^1 = \max u^1(a^1, \alpha^2)$$

or

$$\underline{v}^1 = \min_{\alpha^2 \in BR^2(\alpha^1)} \max u^1(a^1, \alpha^2)$$

that is, constrained minmax

Sample Calculation

	L	M	R
U	0,-3	1,2	0,3
D	0,3*	2,2	0,0

static Nash gives 0

minmax gives 0

worst payoff in fact is 0

pure precommitment also 0

Mixed Precommitment

p is probability of up

to get more than 0 must get SR to play M

$-3p+(1-p)3 \le 2$ and $3p \le 2$

first one

$-3p+(1-p)3 \le 2$

$-3p-3p \le -1$

$p \ge 1/6$

second one

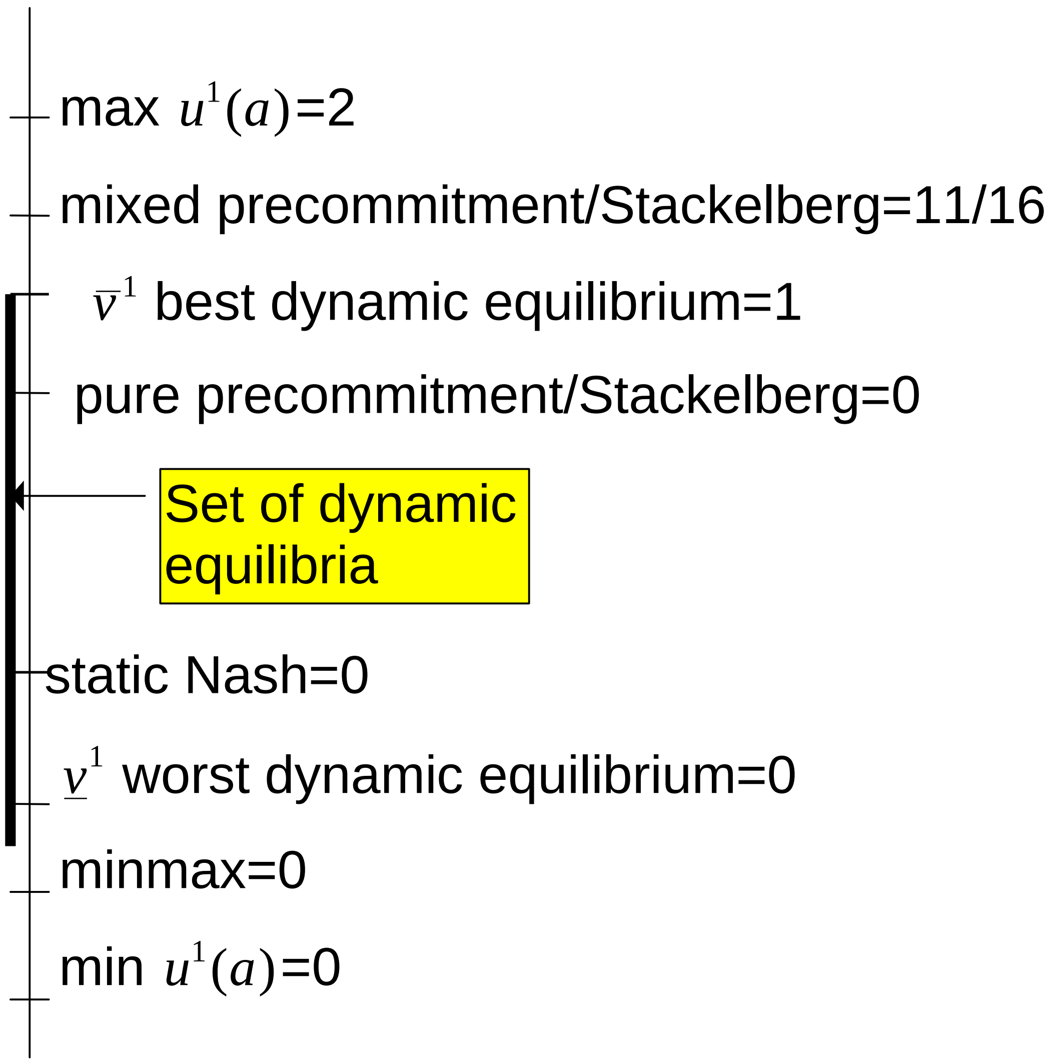
$3p \leq 2$

$p \leq 2/3$

want to play D so take $p = 1/6$

get $1/6 + 10/6 = 11/6$

utility to long-run player



Calculation of best dynamic equilibrium payoff

p is probability of up

p	BR^2	worst in support
<1/6	L	0
1/6<*p<5/6*	M	1
p>5/6	R	0

so best dynamic payoff is 1