Continuous Time Models of Repeated Games with Imperfect Public Monitoring

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What Happens In Repeated Games With Short Periods?

- A common model: continuous time limit
- Two effects in general: player more patient, information less good
- Impact of distribution of signals in a fixed discrete-time game
- Change in distribution with the period length
- Focus on case of long-run versus short-run
- Primarily expositional: see Abreu, Pearce and Milgrom [1991], Sannikov [2006], Sannikov and Skrypcaz [2006], Faingold and Sannikov [2005], Faingold [2005]
- What is the underlying economics of all these results?

Basics – Long Run versus Long Run

Fudenberg, Levine and Maskin [1995] folk theorem

Under mild informational conditions any individually rational payoff vector approximated by equilibrium payoff if common discount factor of the players is sufficiently close to one

Sannikov [2005] characterizes equilibrium payoffs in continuous time where information follows vector valued diffusion, proves a folk theorem when information has a product structure and limit of interest rates $r \to 0$.

Basics – Long Run versus Short Run

Fudenberg and Levine [1994] LP algorithm to compute limit of equilibrium payoffs as discount factor of the long-run players converges to one and characterizes limit payoffs when information has a product structure; typically bounded away highest payoff when all players are long-run, but better than static Nash

Faingold and Sannikov [2005] show set of equilibria in continuous time where information is a diffusion process is only the static equilibrium

Abreu, Pearce and Milgrom [1991] implicitly show that with continous time Poisson information “bad news” signals lead to folk theorem, “good news” signal lead to static Nash

Summary

- Long run versus long run – length of period makes little difference
- Long run versus short run – length of period makes a big difference
  - "good news" Poisson or diffusion leads to static Nash
  - "bad news" Poisson leads to folk theorem

Long-Run versus Short-Run

two-person two-action stage game payoff matrix

	Player 2		
Player 1		**L**	**R**
	+1	$\underline{u}$,0	u,1
	-1	$\underline{u}$,0	$u+g$,-1

$\underline{u} < u, g > 0$

2 plays **L** in every Nash equilibrium

player 1's static Nash payoff $\underline{u}$, also minmax payoff

player 1 prefers that player 2 play **R**

can only induce player to play **R** by avoiding playing **–1**

classic time consistency problem

Information

end of stage game public signal $z \in \mathbb{R}$ observed

depends only on action taken by player 1

(player 2's action publicly observed)

public signal drawn from $F(z \mid a_1)$

F is either differentiable and strictly increasing

or corresponds to discrete random variable

$f(z \mid a_1)$ denotes density function

monotone likelihood ratio condition

$f(z \mid a_1 = -1) / f(z \mid a_1 = +1)$ strictly increasing in z

means that z is "bad news" about player 1's behavior in sense that it means player 1 probably playing **–1**

Other Stuff

Availability of public randomization device

τ length of period

player 1 long-run player with discount factor $\delta = 1 - r\tau$

player 2 an infinite sequence of short-run opponents

Best Perfect Public Equilibrium for LR

largest value v that satisfies incentive constraints

$$v = (1-\delta)u + \delta \int w(z) f(z \mid a_1 = +1) dz$$

$$v \geq (1-\delta)(u+g) + \delta \int w(z) f(z \mid a_1 = -1) dz$$

$$v \geq w(z) \geq \underline{u}$$

or $v = \underline{u}$ if no solution exists

second incentive constraint must hold with equality

otherwise increasing the punishment payoff w retains incentive compatibility and increases utility on the equilibrium path

Cut-Point Equilibria

monotone likelihood ratio condition implies these best equilibria have a cut-point property

$\tilde{z}^*$ is cut point

continuous z: a fixed cut-point

discrete z: a cut-point randomized between two adjacent grid-points

Proposition 1: There is a solution to the LP problem characterizing the most favorable perfect public equilibrium for the long-run player with the continuation payoffs $w(z)$ given by

$$w(z) = \begin{cases} w & z \geq \tilde{z}^* \\ v & z < \tilde{z}^* \end{cases}$$

and indeed, $w = \underline{u}$

Measures of Information

continuous case define

$$p = \int_{z^*}^{\infty} f(z \mid a_1 = +1)dz, q = \int_{z^*}^{\infty} f(z \mid a_1 = -1)dz$$

interested in case in which τ is small

information $q(\tau), p(\tau)$ functions of τ

$\rho, \mu \in \Re \cup \{\infty\}$ *regular values* of $q(\tau), p(\tau)$ if along some sequence $\tau^n \to 0$

$\rho = \lim_{\tau^n \to 0}(q(\tau^n) - p(\tau^n))/p(\tau^n)$ [signal to noise]

$\mu = \lim_{\tau^n \to 0}(q(\tau^n) - p(\tau^n))/\tau^n$ [signal arrival rate]

$v^* =$ [illegible] $\lim_{\tau^n \to 0} \max\{\underline{u}, v^*\}$

(***) $\mu\left(\frac{(u-\underline{u})}{g}-\frac{1}{\rho}\right)$

if positive and $\bar{v}>\underline{u}$ there is a *non-trivial* limit equilibrium

exists positive τ, r such that for all smaller values exists equilibrium giving long-run player more than $\underline{u}$

conversely, if either $\bar{v}\leq\underline{u}$ or (***) is non-positive then for any fixed $r>0$ along the sequence τ^n the best equilibrium for long-run converges to $\underline{u}$

if $\bar{v}=u$ say that limit equilibrium is efficient: if and only if

Proposition 2: Suppose that ρ,μ are regular. Then there is a non-trivial limit equilibrium if and only if $\rho>g/(u-\underline{u})$ *and* $\mu>0$ and $\rho>0$. There is an efficient limit equilibrium if and only if $\mu>0$ and $\rho=\infty$.

Poisson Case

public signal of long-run generated by continuous time Poisson

Poisson arrival rate is

λ_p if action is **+1**

λ_q if action is **–1**

"good news" signal means probably played **+1**: $\lambda_q < \lambda_p$; z number of signals

"bad news" signal means probably played **–1**: $\lambda_q > \lambda_p$; z negative of number of signals

bad-news case $\lambda_q > \lambda_p$

cutoff number of signals before punishment $v - w$

two or more signals isn't interesting since probability of punishment is only of order τ^2

suffices to consider the cutoff in which punishment always occurs whenever any signal is received

probability of punishment $p(\tau) = 1 - e^{-\lambda_p \tau}, q(\tau) = 1 - e^{-\lambda_q \tau}$, as the long-run player plays **–1** or **+1**

then $\rho = (\lambda_q - \lambda_p)/\lambda_p, \mu = \lambda_q - \lambda_p$ (big and positive respectively)

$v^* = u - g\lambda_p /(\lambda_q - \lambda_p)$

note independence of payoff $\underline{u}$

good news" case $\lambda_q < \lambda_p$

punishment triggered by small number of signals, rather than large

if there is punishment, must occur when no signals arrive

probability of punishment when no signal $\gamma(\tau)$

$$p(\tau) = \gamma(\tau)e^{-\lambda_p \tau}, q(\tau) = \gamma(\tau)e^{-\lambda_q \tau}$$

regardless of $\gamma(\tau)$ implies $\rho = 0$, so only trivial limit

Overview

with short run providing incentives to long-run has non-trivial efficiency cost

“good news” case, providing incentives requires frequent punishment

many independent and non-trivial chances of a non-trivial punishment in a small interval of real time, long run player’s present value must be low

contrast, can be non-trivial equilibrium even in the limit when signal used for punishment has negligible probability (as in bad-news case)

<u>or</u> several long run players so punishments can take the form of transfers payments

The Diffusion Case

signals generated by diffusion process in continuous time

drift controlled by the long-run action

sample process at intervals of length τ implies signals have variance $\sigma^2\tau$

we allow the variance signal $\sigma^2\tau^{2\alpha}$ where $\alpha < 1$, with diffusion corresponding to $\alpha = 1/2$

mean of the process is $-a_1\tau$ (recall that $a_1 =$**+1** or **–1**)

so:

$$p = \Phi\left(\frac{-z^* - \tau}{\sigma\tau^\alpha}\right)$$

$$q = \Phi\left(\frac{-z^* + \tau}{\sigma\tau^\alpha}\right)$$

where Φ is standard normal cumulative distribution

Proposition 3: For any $\alpha < 1$ there exists $\underline{\tau} > 0$ such that for $0 < \tau < \underline{\tau}$ there is no non-trivial limit equilibrium

true even when $\alpha > 1/2$, where process converges to deterministic one

contrast "bad news" Poisson case: like diffusion case corresponds to $\alpha = 1/2$

exact form of noise matters: is it a series of unlikely negative events, as in the "bad news" Poisson case, or a sum of small increments as in the normal case?

contrast the diffusion case $\alpha = 1/2$ with a sum of small increments where the scale of the increment is proportional to the length of the interval

standard error of the signal of order τ

corresponds to case $\alpha = 1$

take the limit of such sequence of processes, limit is deterministic process without noise.

Proposition 4: If $\alpha = 1$ there exists $\underline{\tau}$ such that for all $0 < \tau < \underline{\tau}$ (*) is satisfied, and $\lim_{\tau \to 0} v^* = u$.

for fixed τ taking a very large cutoff $z^* \to \infty$

causes the likelihood ratio $q/p \to \infty$, so $p/(q-p) \to 0$

so $v^* \to 1$

note $p, q \to 0$, so for fixed $\underline{u}$ and τ and z^* sufficiently large, (*) must be violated

for any choice of z^*, r, τ, there always $\underline{u}$ sufficiently small that (*) holds

worst punishment determines the best equilibrium

going far enough into tail of normal, arbitrarily reliable information can be found about whether a deviation occurred

information can be used to create incentives, provided sufficiently harsh punishment available

when $\alpha = 1$ signal to noise ratio improves sufficiently quickly that we can exploit the shorter intervals to choose a bigger cutoff value of ζ

Limits of Poisson Processes

consider for examples “sales” or “revenues”

made up of sum many individual transactions

consider small enough time intervals observe at most a single transaction

so Poisson processes seem natural for studying economic signals

in many circumstances, individual transactions not visible suppose that public signal of long-run player's action generated by observing an underlying Poisson process in continuous time

Poisson arrival rate:

λ_p if the action **+1**

λ_q if the action **–1**

observe number of Poisson events (or in "good news" case its negative) fix the Poisson process generating "sales"

conclude as above that small time interval get a non-degenerate limit if and only if events correspond to "bad news"

this model implies that only individual transactions are observed

allow the Poisson parameters to vary with time interval

$\lambda_p(\tau), \lambda_q(\tau)$

expected number of events over interval of length τ is $\lambda_p(\tau)\tau, \lambda_q(\tau)\tau$.

consider case where expected number of events per period very large – that is $\lambda_q(\tau)\tau, \lambda_p(\tau)\tau \to \infty$

from Central Limit Theorem Poisson random variable y has $(y - Ey)/(\mathrm{var}\, y)^{1/2}$ approach a standard normal

claim: good news versus bad news doesn't make any difference once we start aggregating

similar to Hellwig-Schmidt