Expected Utility Theory

Let Ω be a probability space

A gamble is a random variable where the quantity represents "money" or "consumption"

Suppose that x_1 and x_2 are "gambles"

Which gamble is preferred?

Generally: gains are less important than losses

Von Neumann-Morgenstern Preferences

Gambles are compared using a numeric valued utility function

$u(x)$ is the utility from consuming *x*

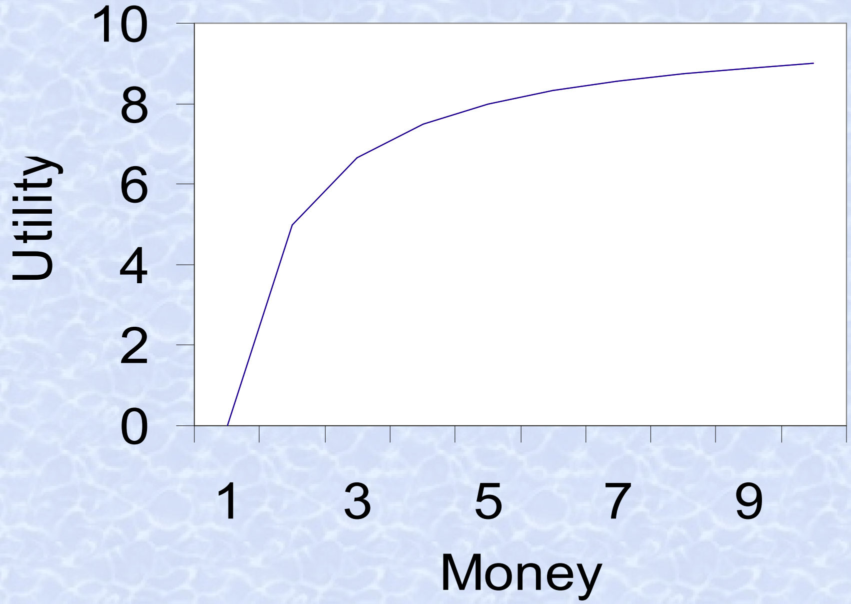
x_1 is at least as good (strictly better than) as x_2

$Eu(x_1) \geq (>) Eu(x_2)$

risk neutrality: $u(x) = x$

Example

$$u(x) = 10 - 10 / x$$



Money versus Utility

50-50 Heads vs Tails; two gambles Up vs Down

Money payoffs for player 1

	H	T
U	5	1
D	4	2

Utility payoffs for player 1

	H	T
U	8	0
D	7.5	5

Optimal Choices

If H and T have equal probability is it better to choose U or D?

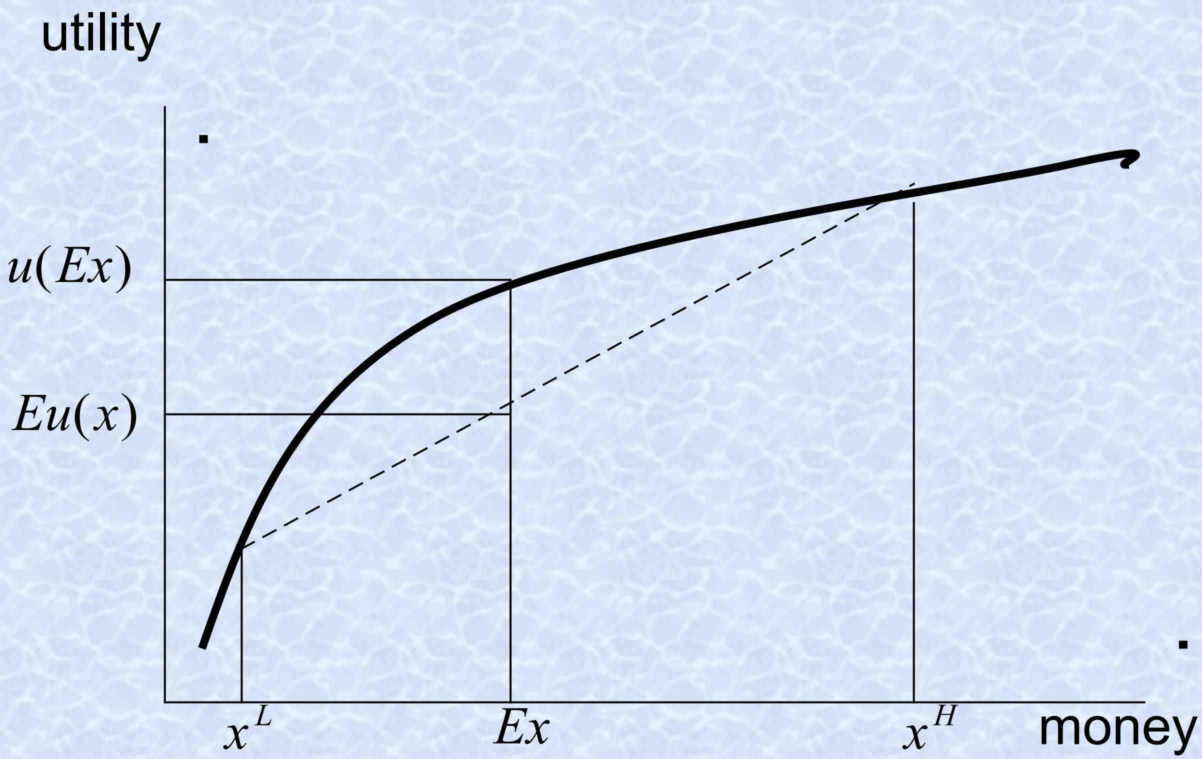
	Expected money	Expected utility
U	3	4
D	3	6.25

Choose D

Risk Aversion

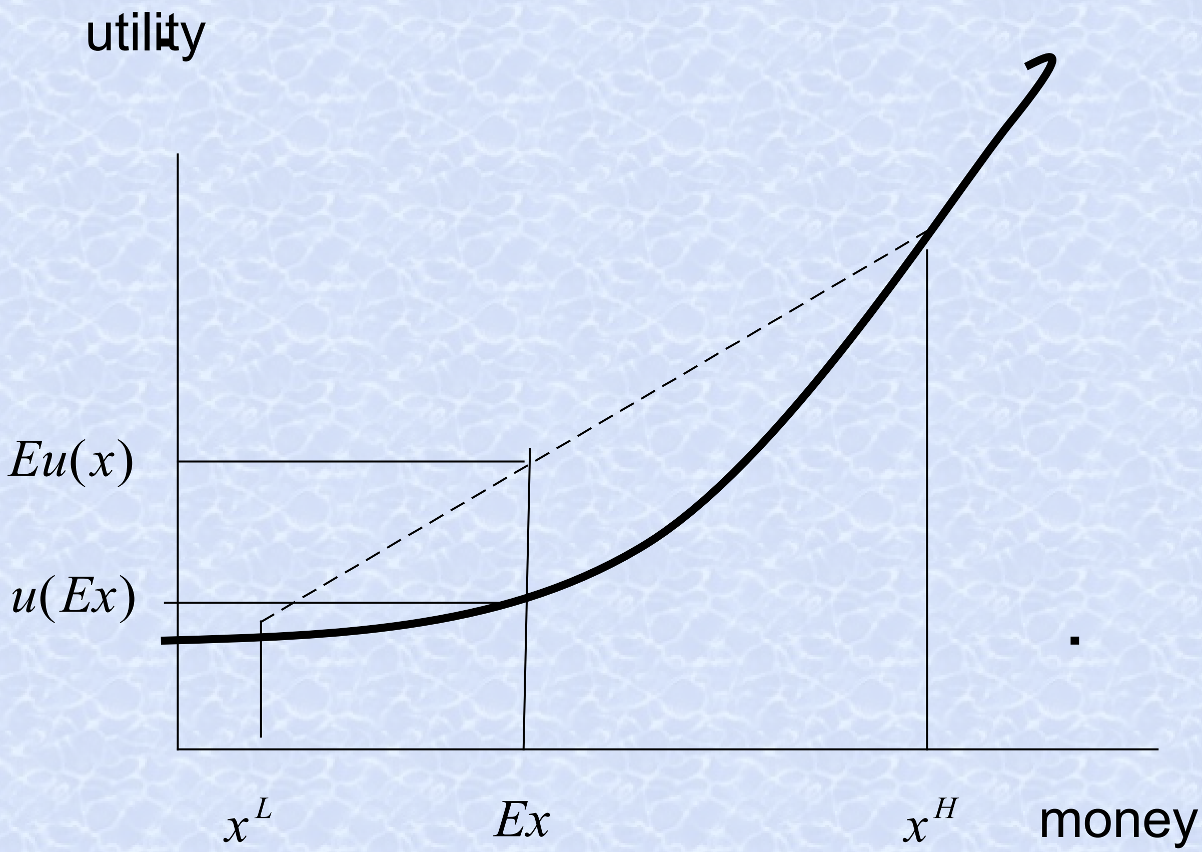
Would you rather get a gamble *x* or get the expected value of the gamble E*x* for sure? Suppose that the gamble is x^L with probability *p* and x^H with probability 1-*p*

Concavity



What happens as *p* changes?

Risk Loving



Knee Breakers

risk loving because the loss function is truncated

- you have 1000 and owe a gambling debt of 2000
- double or nothing is a good bet: lose and you still get your knees broken; win and you escape

other applications:

- sporting contests: “the Hail Mary pass”
- the game of banks and regulators

Applications

- Investment: risky portfolio? Stocks or bonds?
- Insurance: auto insurance company charges a premium
    - diversification
    - but hard risk difficult to insure
    - example: industrial decline, everyone should pay, but how to get them to do it?
    - some economists do not really understand competition and see *market failure* everywhere
    - some “libertarian” non-economists see market failure nowhere
    - in fact market failure is a problem with insurance for large risks

Certainty Equivalent

y a random amount with $Ey = 0, Ey^2 = 1$

the wealth of endowment is x

the gamble is $\sigma x y$ where σ measure the size of the gamble

the gamble is proportional to wealth

the idea of *certainty equivalent* $\overline{x}$ is the certain amount that is equivalent to the gamble

$$u(\overline{x}) = u(x + \sigma x y)$$

with risk aversion this is smaller than the endowment

$\overline{x} < x$

Relative Risk Premium

relative risk premium π is the proportional difference between endowment and certainty equivalent

$$u - \pi x = \overline{x}$$

$$u(x - \pi x) = u(x + \sigma x y)$$

$$u(x) - \pi x u'(x) = Eu(x) + \sigma x u'(x) y + (1/2)\sigma^2 x^2 u''(x) y^2$$

$$= u(x) + (1/2)\sigma^2 x^2 u''(x)$$

$$\pi = (1/2)\rho\sigma^2$$

where ρ is the *coefficient of relative risk aversion*

$$\rho = -\frac{u"(x)x}{u'(x)}$$

important: the premium increases as the ***square*** of the scale of the gamble σ

Constant Relative Risk Aversion

$u(x) = \frac{x^{1-\rho}}{1-\rho}$ also known as "constant elasticity of substitution" or CES

$\rho \geq 0$

compute the coefficient of relative risk aversion

$$-\frac{u''(x)x}{u'(x)} = \frac{\rho x^{-\rho-1}x}{x^{-\rho}} = \rho$$

$\rho = 0$ linear, risk neutral

$\rho = 1$ $u(x) = \log(x)$

useful for empirical work and growth theory, perhaps ρ is about two?

Example

Logarithmic utility, good approximation in many circumstances

$u(x) = \log \mathrm{x}$

endowment: $x_0 = 100$

two investments of 10

stock: 75% gain of 20, 25% no gain

bond: certain gain of 12

	utility	
endowment	$\log 100$	4.605
stock	$.75 \log 110 + .25 \log 90$	4.650
bond	$\log 102$	4.625

What if?

Endowment: $x_0 = 1000$

two investments of 100

stock: 75% gain of 200, 25% no gain

bond: certain gain of 120

Concepts

- **expected utility**
- **risk aversion**, risk loving, **risk neutral**
- insurance, market failure
- concavity and convexity
- risk premium, certainty equivalent
- **coefficient of relative risk aversion**

Skill

given different investments with different risky returns and a constant relative risk aversion utility function

- find which is the superior investment
- determine how the answer depends upon risk aversion